INTRODUCTION TO THE THEORY

OF NEUTRON DIFFUSION

by

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Numerical work by Bengt Carlson and Max Goldstein

PREFACE

This volume is an introduction to work on the theory of neutron diffusion carried out under the auspices of the British, Canadian, and United States wartime atomic energy projects and the U. S. Atomic Energy Commission.

The subject matter is based on a series of lectures delivered by George Placzek during the summer of 1949 at the Rand Corporation, Santa Monica, California, and at the University of California at Los Angeles. These lectures were sponsored by the Los Alamos Scientific Laboratory.

Frederic de Hoffmann recorded the lectures and from these prepared a first draft of this volume. K. M. Case, aided by George Placzek, carried out the major work of amplifying and adding new material to the book. Thus the final version of this volume was written by K. M. Case. Numerical work presented here is based partly on wartime reports but is largely computational work done since 1949 at the Los Alamos Scientific Laboratory. These computations were carried out under the supervision of Bengt Carlson and Max Goldstein.

We wish to thank the Rand Corporation for its hospitality in the summer of 1949 and express our appreciation to Ralph Carlisle Smith of the Los Alamos Scientific Laboratory for his interest and help.

Our thanks are also due to Helen Stearns, Jane Amenta and Louis Jacot for their help in preparing the manuscript.

<u>CONTENTS</u>

A. INTRODUCTION

The object of the theory to be discussed here is the determination of the distribution of neutrons in space and time in terms of the geometrical configuration and the physical properties of the medium. We will limit ourselves to one velocity problems in which it is assumed that the magnitude of the neutron velocity is unchanged on collision.

Application of this theory to the diffusion of thermal neutrons implies neglect of the velocity spread. Similarly the one velocity treatment of fast neutron problems omits slowing down effects. These approximations are seldom fully justifiable. In view of this fact, it might appear that a very detailed development of the one velocity theory is of somewhat doubtful value. However, such a conclusion would be incorrect. Certainly the one velocity problem must be solved before one can hope to include the effects of changes of the magnitude of the velocity. Moreover, one velocity theory can be used directly for the solution of more general problems. Thus the problem of slowing down by elastic collisions can be reduced to a one velocity problem by taking a Laplace Transform in the energy. The Laplace Inversion of the solution of the one velocity problem can be successfully carried out despite the dependence of the scattering law on the transform variable.

In addition, the general question of the diffusion of neutrons with energy change may be treated by dividing the neutrons into velocity groups. The neutrons in each group will then be described by one velocity equations with sources which depend on the neutrons of the other groups.

Part B of the discussion is of a preparatory nature. After describing streaming in vacuum, the motion of neutrons in absorbing non-scattering media is considered. Though rather trivial, this part does have practical applications for the treatment of problems involving holes and pure absorbers. Of somewhat greater importance is the opportunity it affords to introduce analytical methods used in Part C in a particularly simple and obvious form. Moreover, many of the formal results obtained in B can be carried over directly to C.

In Part C we come to the general problem of one velocity neutron diffusion.

1. Definitions

The neutron distribution is characterized by the angular density $\psi(\vec{r}, \vec{\Omega}, t)$, the number of neutrons per unit volume and unit solid angle moving in the direction of the unit vector $\vec{\Omega}$. $\vec{r}$ and t denote the space coordinates and the time, respectively. Directly related to ψ is the angular current $v\vec{\Omega}\psi(\vec{r}, \vec{\Omega}, t)$. The magnitude of this vector is the number of neutrons crossing unit area perpendicular to the direction of $\vec{\Omega}$ per unit time and unit solid angle.

The integrals of the angular density and angular current over all directions $\vec{\Omega}$ are the density ρ, the number of neutrons per unit volume, and the current $\vec{j}$, respectively. The magnitude of the vector $\vec{j}$ is the net number of neutrons crossing a unit area perpendicular to $\vec{j}$ per unit time. Thus:

$$\rho(\vec{r}, t) = \int \psi(\vec{r}, \vec{\Omega}, t)\, d\Omega$$

$$\vec{j}(\vec{r}, t) = v\int \vec{\Omega}\psi(\vec{r}, \vec{\Omega}, t)\, d\Omega$$

If neutrons are being created in the medium the determination of ψ requires knowledge of the source distribution. This source distribution will be characterized by the angular source density $q(\vec{r}, \vec{\Omega}, t)$ which represents the number of neutrons of direction $\vec{\Omega}$ produced per unit volume, unit time, and unit solid angle. The <u>source density</u> $q_o(\vec{r}, t)$ is the integral of q over all directions.

i.e.,

$$q_o(\vec{r}, t) = \int q(\vec{r}, \vec{\Omega}, t)\, d\Omega$$

B. PROPAGATION IN THE ABSENCE OF SCATTERING COLLISIONS

I. Streaming in Vacuum

2. Continuity Equation

In vacuum the angular density ψ must satisfy the continuity equation

$$\frac{\partial \psi}{\partial t}(\vec{r},\vec{\Omega},t) + \text{div}\left(v\vec{\Omega}\psi(\vec{r},\vec{\Omega},t)\right) = q(\vec{r},\vec{\Omega},t) \tag{1}$$

which states that the time rate of change of the angular density is equal to the angular source density minus the number of neutrons leaving unit volume per unit time.

For the stationary case $\frac{\partial \psi}{\partial t} = 0$ and (1) reduces to

$$\text{div}\left(v\vec{\Omega}\,\psi(\vec{r},\vec{\Omega})\right) = q(\vec{r},\vec{\Omega}) \tag{2}$$

which, since v has been assumed constant, may be written:

$$v\vec{\Omega}\cdot\text{grad}\ \psi(\vec{r},\vec{\Omega}) = q(\vec{r},\vec{\Omega}) \tag{3}$$

3. Source-Free Streaming

In the absence of sources (2-3) reduces to

$$\Omega\cdot\text{grad}\ \psi = 0 \tag{1}$$

or

$$\frac{\partial \psi}{\partial s} = 0 \tag{2}$$

where ds is the line element along $\vec{\Omega}$. $\psi(\vec{r},\vec{\Omega})$ is constant along any line parallel to $\vec{\Omega}$.

For a region bounded by a closed surface (Fig. 1) ψ is the same at the points A, B and C which lie along a line in the direction $\vec{\Omega}$.

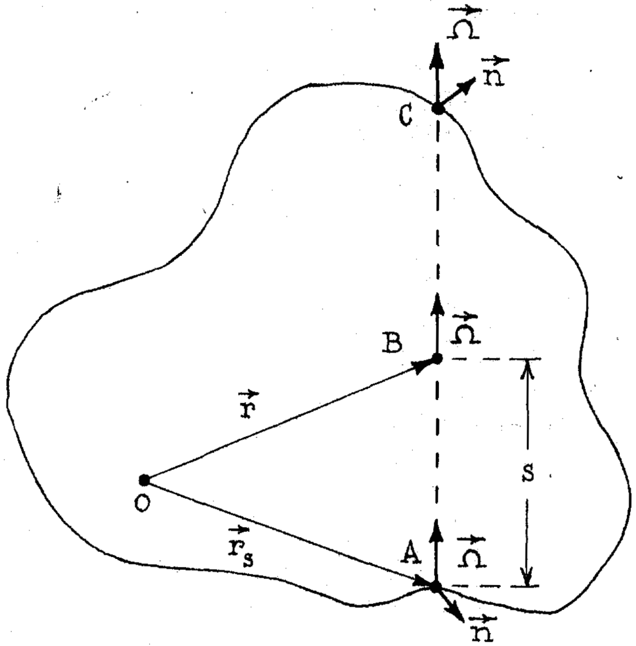

Fig. 1

Thus the incoming angular density at A is equal to the outgoing angular density at C and only one of these angular densities is to be specified to give a complete boundary condition. If $\vec{r}_s$ be the point on the boundary which lies in the direction $-\vec{\Omega}$ from $\vec{r}$ and the incoming angular density at $\vec{r}_s\left(\psi_S(\vec{r}_s, \vec{\Omega})\right)$ be given, equation (2) has the solution

$$\psi(\vec{r}, \vec{\Omega}) = \psi_S(\vec{r}_s, \vec{\Omega}) = \psi_S(\vec{r} - s\vec{\Omega}, \vec{\Omega}) \tag{3}$$

(here s is the distance between $\vec{r}$ and $\vec{r}_s$.)

If ψ_S is independent of $\vec{r}_s$; i.e.,

$$\psi_s(\vec{r}_s, \vec{\Omega}) = f(\vec{\Omega}) \tag{4}$$

then by (3)

$$\psi(\vec{r}, \vec{\Omega}) = f(\vec{\Omega}) \tag{5}$$

Thus the interior angular density is also position independent. In particular if the distribution on the boundary be isotropic it will also be so within the region.

To illustrate the use of the solution (3) of (1) let us consider some simple cases with spherical symmetry. Since with such symmetry

$\psi(\vec{r}, \vec{\Omega})$ can only depend on the radial distance r and the angle between $\vec{\Omega}$ and $\vec{r}$, it is convenient to write

$$\psi(\vec{r}, \vec{\Omega}) = \frac{1}{2\pi} n(r, \mu) \tag{6}$$

where

$$\mu = \frac{\vec{r}}{r} \cdot \vec{\Omega} \tag{7}$$

Then from the original definitions of density and current we have:

$$\rho(r) = \int_{-1}^{1} n(r, \mu)\, d\mu \tag{8}$$

$$\vec{j}(r) = v\left(\frac{\vec{r}}{r}\right)\int_{-1}^{1} \mu n(r, \mu)\, d\mu \tag{9}$$

(9) states that the current is directed along the radius vector and its magnitude is only a function of r.

Integrating (1) over $\vec{\Omega}$ gives the conservation equation

$$\operatorname{div} \vec{j} = 0 \tag{10}$$

which reduces for spherical symmetry to

$$\frac{1}{r^2} \frac{d}{dr} (r^2 j) = 0 \tag{11}$$

and hence

$$j = c/4\pi r^2 \tag{12}$$

where the constant c is the number of neutrons flowing out of any sphere of radius "a." The angular density as a function of r and μ is given by (3) which in the new variables is:

$$n(r, \mu) = n(a, \mu_s) \tag{13}$$

where

$$\mu_s = \left(\frac{\vec{r}}{r}\right)_{r = s} \cdot \vec{\Omega} \tag{14}$$

Thus μ_s is determined as follows:

Draw the radius vector to the point $\vec{r}$ and the unit vector $\vec{\Omega}$ at $\vec{r}$ which is such that $\vec{\Omega}\cdot\frac{\vec{r}}{r} = \mu$. Extend $\vec{\Omega}$ back until it intersects the sphere $r = a$. The cosine of the angle between $\vec{\Omega}$ and the radius vector at the intersection is μ_s. Figures 2 and 3 show this construction for $r < a$ and $r > a$, respectively.

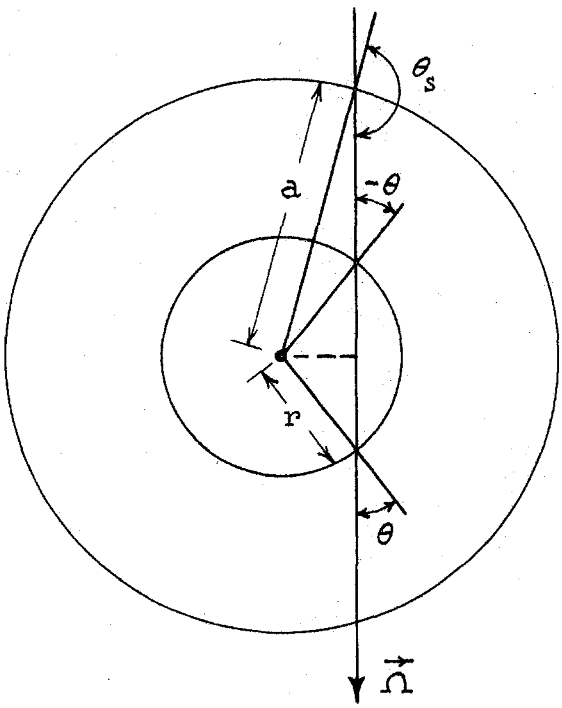

Fig. 2

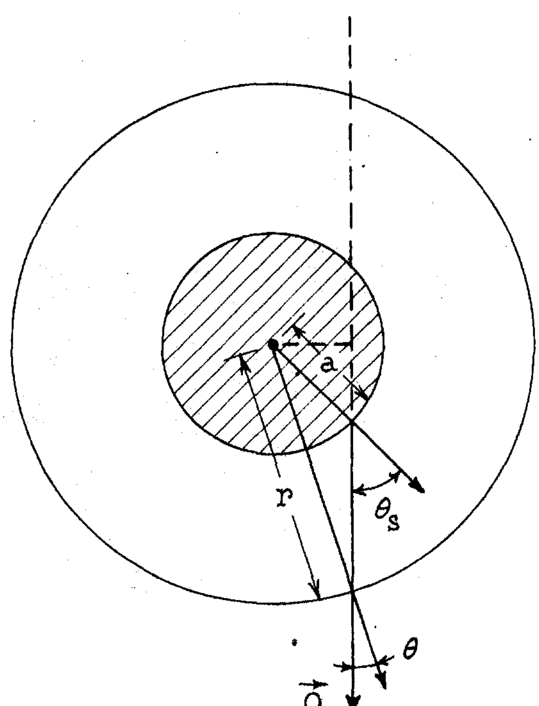

Fig. 3

We see that $n(r,\mu)$ for $r < a$ is determined by values of $n(a, \mu_s)$ in the range $-1 \leq \mu_s \leq 0$ while $n(r, \mu)$ for $r > a$ is determined by $n(a, \mu_s)$ in the range $0 \leq \mu_s \leq 1$.

Let us first consider the case $r < a$. Since $\mu = \cos\theta$, $\mu_s = \cos\theta_s$ and we see from Fig. 2 that

$$a \sin\theta_s = r \sin\theta \tag{15}$$

We have

$$\mu_s = -\sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)} \tag{16}$$

(16) shows that the values of μ_s relevant for the determination of $n(r,\mu)$ lie in the range $-1 \leq \mu_s \leq -\sqrt{1 - \frac{r^2}{a^2}}$. Neutrons leaving the sphere of radius "a" at angles with larger values of μ_s will not approach within a distance r of the origin.

Solving (16) for μ gives:

$$\mu = \pm \sqrt{1 - \frac{a^2}{r^2}(1 - \mu_s^2)} \tag{17}$$

The ambiguity in sign in (17) is due to the fact that neutrons leaving "a" with a given μ_s contribute to $n(r,\mu)$ for two values of μ, corresponding to the point at which they enter the sphere of radius r and to the point at which they leave.

The angular density within the sphere is given explicitly by inserting (16) into (13):

$$n(r,\mu) = n\left(a, -\sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)}\right) \tag{18}$$

(18) gives on noting that $n(r,\mu)$ is an even function of μ :

$$\rho(r) = 2\int_0^1 n(r,\mu)\, d\mu = 2\int_0^1 n_s(\mu_s)\, d\mu \tag{19}$$

As examples let us carry through the computation for the three cases $n(a, \mu_s) = 1; \ -\mu_s; \ -\frac{1}{\mu_s}$

For $n(a, \mu_s) = 1$:

$$n(r, \mu) = n(a, \mu_s) = 1 \tag{20}$$

and

$$\rho(r) = 2 \tag{21}$$

For $n(a, \mu_s) = -\mu_s$:

$$n(r, \mu) = -\mu_s = +\sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)} \tag{22}$$

and

$$\rho(r) = 2\int_0^1 \sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)} \, d\mu = 1 + \left(1 - \frac{r^2}{a^2}\right) \frac{\tanh^{-1}(\frac{r}{a})}{(\frac{r}{a})} \tag{23}$$

We see that $\rho(o) = 2$ and $\rho(a) = 1$ which gives evidence of a focusing effect due to the preference for the radial direction of this incident distribution.

For $n(a, \mu_s) = -\frac{1}{\mu_s}$

$$\rho(r) = 2\int_0^1 \frac{d\mu}{\sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)}} = 2\,\frac{\tanh^{-1}(\frac{r}{a})}{(\frac{r}{a})} \tag{24}$$

In particular $\rho(o) = 2$ and $\rho(a) = \infty$, showing the way the incident distribution emphasizes the lateral directions.

For $r > a$ we find, using Fig. 3, that equation (18) still holds and that

$$\mu_s = +\sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)} \tag{25}$$

or $$\mu = + \sqrt{1 - \frac{a^2}{r^2}(1 - \mu_s^2)} \tag{26}$$

Here μ_s may range from 0 to 1. The corresponding values of μ for which $n(r,\mu)$ does not vanish are $\sqrt{1 - \frac{a^2}{r^2}} \leqslant \mu \leqslant 1$. Thus for the density we have

$$\rho(r) = \int_{\sqrt{1 - \frac{a^2}{r^2}}}^{1} n(r,\mu)\, d\mu \tag{27}$$

For the three simple cases previously considered we find:

For $n(a, \mu_s) = 1$

$$\rho(r) = \int_{\sqrt{1 - \frac{a^2}{r^2}}}^{1} d\mu = 1 - \sqrt{1 - \frac{a^2}{r^2}} \tag{28}$$

It should be pointed out that the density is not constant despite the isotropic boundary condition, in apparent contradiction with (5). This contradiction, however, is resolved by noting that the bounding surface is here not closed.

For $n(a, \mu_s) = \mu_s$

$$\rho(r) = \int_{\sqrt{1 - \frac{a^2}{r^2}}}^{1} \sqrt{1 - \frac{r^2}{a^2}(1 - \mu^2)}\, d\mu \tag{29}$$

$$= \frac{1}{2}\left[1 - \left(1 - \frac{a^2}{r^2}\right)\frac{\tanh^{-1}\frac{a}{r}}{\left(\frac{a}{r}\right)}\right]$$

For $n(a, \mu_s) = \frac{1}{\mu_s}$

$$\rho(r) = \frac{a^2}{r^2} \int_0^1 \frac{1}{\sqrt{1 - \frac{a^2}{r^2}(1 - \mu_s^2)}} \, d\mu_s \tag{30}$$

$$= \frac{a}{r} \tanh^{-1} \frac{a}{r}$$

The difference of a factor of two between the $\lim_{r \to a+} \rho(r)$ and $\lim_{r \to a-} \rho(r)$ for the same surface distribution arises from the fact that the internal distribution receives two contributions from the surface (opposite sides) while the external distribution receives only one contribution.

4. Sources

For steady state problems with sources present the equation of continuity (2-2) may be written

$$v \frac{\partial \psi}{\partial s} (\vec{r}, \vec{\Omega}) = q(\vec{r}, \vec{\Omega}) \tag{1}$$

which has the general solution

$$\psi (\vec{r}, \vec{\Omega}) = \int \frac{q ds}{v} + \psi_h(\vec{r}, \vec{\Omega}) \tag{2}$$

Here ψ_h is a solution of the homogeneous equation (3-1) and the integral is to be taken along a line in the direction $\vec{\Omega}$ up to the point $\vec{r}$.

If neutrons are being produced on a surface it is convenient to define an <u>angular surface source density</u> q_s which is the number of neutrons produced per unit area, unit time and unit solid angle moving in the direction $\vec{\Omega}$. q_s may be related to an angular source density q by the relation

$$q(\vec{r},\vec{\Omega})\, d\Omega = q_s(\vec{r},\vec{\Omega})\ \delta(x)\, d\Omega \tag{3}$$

where x is a coordinate in the direction perpendicular to the surface at $\vec{r}$.

Then if only the surface source q_s be present (1) becomes:

$$v \frac{\partial \psi}{\partial s} = q_s\ \delta(x) \tag{4}$$

If $\vec{n}$ denotes the normal to the surface in the direction of increasing x

$$\frac{\partial \psi}{\partial s} = \vec{\Omega}\cdot\vec{n}\ \frac{\partial \psi}{\partial x} \tag{5}$$

and (3) becomes

$$v\,\vec{\Omega}\cdot\vec{n}\ \frac{\partial \psi}{\partial x} = q_s\ \delta(x) \tag{5}$$

with the solution

$$v\vec{\Omega}\cdot\vec{n}\ (\psi_+ - \psi_-) = q_s \tag{7}$$

where the subscripts + and - distinguish ψ on the plus and minus sides of the surface. (7) shows that a surface source is equivalent to a discontinuity q_s of the normal component of the angular current. If in particular all neutrons come from the surface source $\psi_+(\vec{r}_s,\vec{\Omega}) = 0$ for $\vec{\Omega}\cdot\vec{n} < 0$ and $\psi_-(\vec{r}_s,\vec{\Omega}) = 0$ for $\Omega\cdot n > 0$. In either case

$$\psi_s = \frac{q_s}{v|\vec{\Omega}\cdot\vec{n}|} \tag{8}$$

showing that the presence of the surface source q_s is equivalent to placing a condition on ψ at the surface.

To illustrate let us consider sources distributed on the surface of a sphere with spherical symmetry. In analogy with the definition (3-6) it is convenient to define a source density $\overline{q_s}(\mu)$ e.g.:

$$q_s(\vec{\Omega}) = \frac{1}{2\pi}\,\overline{q}_s(\mu) \tag{9}$$

and then (8) becomes

$$n(a,\mu) = \overline{q}_s(\mu)/v|\mu| \tag{10}$$

For an isotropic source (10) shows that $n(a,\mu) \sim \frac{1}{|\mu|}$. Hence, due to the divergence at $\mu = 0$, $\rho(a)$ is infinite.

Using (10) the surface sources corresponding to the previously considered surface distributions $n(a,\mu) \sim 1; |\mu_s|; \frac{1}{|\mu_s|}$ may be obtained readily.

Thus if $n(a,\mu)$ = constant = c, (10) gives

$$\frac{\overline{q}_s(\mu)}{v} = c|\mu| \tag{11}$$

If this shell source be normalized so that $q_s(\vec{\Omega})/v$ integrated over all angles $\vec{\Omega}$ and over the surface be unity we obtain:

$$\int \frac{q_s(\vec{\Omega})}{v}\, d\Omega\, ds = \int_{-1}^{1} \frac{\overline{q}_s(\mu)}{v}\, d\mu\, ds = 1 = 4\pi a^2 c \tag{12}$$

or

$$c = 1/4\pi a^2 \tag{13}$$

Therefore:

$$\overline{q}_s(\mu)/v = |\mu|/4\pi a^2 \quad \text{and} \quad n(a,\mu) = \frac{1}{4\pi a^2} \tag{14}$$

For the interior of the sphere (14) combined with (3-18) gives:

$$n(r,\mu) = 1/4\pi a^2 \tag{15}$$

With (15) and the definition (3-8) $\rho(r)$ may be obtained.

For the case $r > a$ we see on referring to Fig. 4 that since the source radiates both inward and outward the point A receives contributions from the sources along $\vec{\Omega}$ at both B and C.

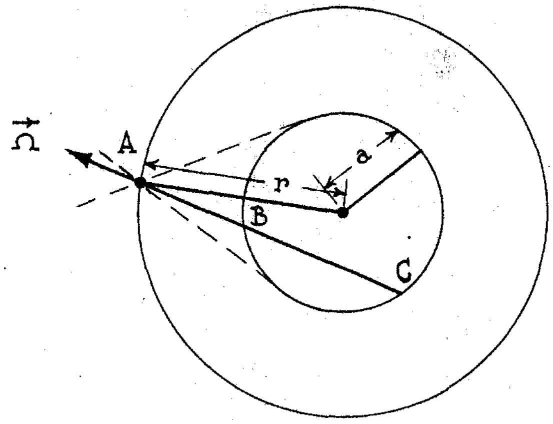

Fig. 4

Hence if

$$\mu > \sqrt{1 - \frac{a^2}{r^2}} \qquad n(r,\mu) = \frac{1}{2\pi a^2} \qquad r > a \tag{16}$$

while if

$$\mu < \sqrt{1 - \frac{a^2}{r^2}} \qquad n(r,\mu) = 0 \qquad r > a \tag{17}$$

since no points of the source contribute to these directions. $\rho(r)$ is again obtained using (3-8).

In Table 1 these results are listed along with those for $n(a,\mu_s) \sim |\mu_s|$ and $1/|\mu_s|$.

Returning to the case of volume sources we note that the general solution (2) of the continuity equation may be written

$$\psi(\vec{r},\vec{\Omega}) = \frac{1}{v}\int_0^\infty q(\vec{r} - R\vec{\Omega},\vec{\Omega})\, dR \tag{18}$$

where the symbols refer to Fig. 5

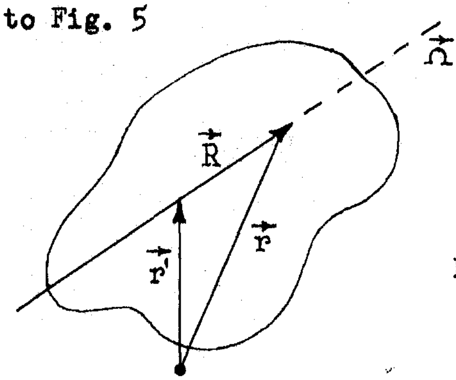

Fig. 5

Here the vector $\vec{R} = R\vec{\Omega}$ connects source and field point. The extension of the integration to infinity permits the omission of the homogeneous

$$\frac{\overline{q}_s(\mu)}{2\pi} = q_s(\vec{\Omega})$$

TABLE 1

Neutron Density Distribution for Various Shell Sources

$$Q = \iint \overline{q}_s(\mu)\, d\mu\, S = \iint q_s(\vec{\Omega})\, d\Omega\, dS$$

$\frac{\overline{q}_s(\mu)}{Q}$	$vn(r,\mu)/Q$		$v\rho(r)/Q$		$\frac{v\rho(o)}{Q}$	$\frac{v\rho(a)}{Q}$
	$r < a$	$r > a$ and $\mu \geq \sqrt{1-\frac{a^2}{r^2}}$	$r < a$	$r > a$		
$\frac{3}{8\pi a^2}\mu^2$	$\frac{3}{8\pi a^2}\left(\frac{r}{a}\right)\sqrt{\mu^2+\frac{a^2}{r^2}-1}$	$\frac{3}{4\pi a^2}\left(\frac{r}{a}\right)\sqrt{\mu^2+\frac{a^2}{r^2}-1}$	$\frac{3}{8\pi a r}\left[\frac{r}{a}+\left(1-\frac{r^2}{a^2}\right)\tanh^{-1}\frac{r}{a}\right]$	$\frac{3}{8\pi a r}\left[\frac{r}{a}+\left(1-\frac{r^2}{a^2}\right)\tanh^{-1}\frac{a}{r}\right]$	$\frac{3}{4\pi a^2}$	$\frac{3}{8\pi a^2}$
$\frac{1}{4\pi a^2}\lvert\mu\rvert$	$\frac{1}{4\pi a^2}$	$\frac{1}{2\pi a^2}$	$\frac{1}{2\pi a^2}$	$\frac{1}{2\pi a^2}\left[1-\sqrt{1-\frac{a^2}{r^2}}\right]$	$\frac{1}{2\pi a^2}$	$\frac{1}{2\pi a^2}$
$\frac{1}{8\pi a^2}$	$\frac{1}{8\pi a^2}\left(\frac{a}{r}\right)\frac{1}{\sqrt{\mu^2+\frac{a^2}{r^2}-1}}$	$\frac{1}{4\pi a^2}\left(\frac{a}{r}\right)\frac{1}{\sqrt{\mu^2+\frac{a^2}{r^2}-1}}$	$\frac{1}{4\pi a r}\tanh^{-1}\left(\frac{r}{a}\right)$	$\frac{1}{4\pi a r}\tanh^{-1}\left(\frac{a}{r}\right)$	$\frac{1}{4\pi a^2}$	∞

solution since this may always be regarded as due to fictitious sources exterior to the region of consideration. Thus considering q within the region to be the given sources and q outside determined so that ψ satisfies the requisite boundary conditions (18) becomes the general solution of (1).

The line integral in (18) may be converted into a volume integral by use of the angular delta function $\delta_2(\vec{\Omega}\cdot\vec{\Omega}')$ defined by

$$\delta_2(\vec{\Omega}\cdot\vec{\Omega}') = 0 \qquad \vec{\Omega} \neq \vec{\Omega}' \tag{19}$$

and

$$\int \delta_2(\vec{\Omega}\cdot\vec{\Omega}')\, d\Omega = \int \delta_2(\vec{\Omega}\cdot\vec{\Omega}')\, d\Omega' = 1 \tag{20}$$

It should be noted that the integration in (20) is only over the direction of the unit vector $\vec{\Omega}$ or $\vec{\Omega}'$. The connection with the usual delta function is expressed by

$$\delta_2(\vec{\Omega}\cdot\vec{\Omega}') = \frac{1}{2\pi}\,\delta(\vec{\Omega}\cdot\vec{\Omega}' - 1) \tag{21}$$

With $\vec{\Omega}'$ chosen in the direction $\vec{R}$ (18) may be written

$$\begin{aligned} \psi(\vec{r},\vec{\Omega}) &= \frac{1}{v}\int \delta_2\left(\vec{\Omega}\cdot\frac{\vec{R}}{R}\right) d\vec{\Omega}_R \int_0^\infty q\left(\vec{r}-\vec{R},\frac{\vec{R}}{R}\right) dR \\ &= \frac{1}{v}\int q\left(\vec{r}-\vec{R},\frac{\vec{R}}{R}\right) \delta_2\left(\vec{\Omega}\cdot\frac{\vec{R}}{R}\right) \frac{d\vec{R}}{R^2} \\ &= \frac{1}{v}\int q\left(\vec{r}',\frac{\vec{R}}{R}\right) \delta_2\left(\vec{\Omega}\cdot\frac{\vec{R}}{R}\right) \frac{d\vec{r}'}{R^2} \end{aligned} \tag{22}$$

Here $d\vec{r}'$ denotes the volume element and the integral is extended over all space.

The density ρ and current $\vec{j}$ may therefore be expressed as

$$\rho = \frac{1}{v}\int \frac{q\left(\vec{r}',\frac{\vec{R}}{R}\right)}{R^2}\, d\vec{r}' \tag{23}$$

$$\vec{j} = \int \frac{\vec{R}}{R}\,\frac{q\left(\vec{r}',\frac{\vec{R}}{R}\right)}{R^2}\, d\vec{r}' \tag{24}$$

For a point source

$$q(\vec{r}', \vec{\Omega}) = Qf(\vec{\Omega}) \; \delta(\vec{r}') \tag{25}$$

which gives on inserting in (22)

$$\psi(\vec{r}, \vec{\Omega}) = \frac{Qf(\frac{\vec{r}}{r}) \; \delta_2(\vec{\Omega} \cdot \frac{\vec{r}}{r})}{vr^2} \tag{26}$$

This is a highly singular function expressing the obvious fact that an observer at point $\vec{r}$ looking along a line $\vec{\Omega}$ which does not pass through the source sees no neutrons. From (23) and (24) we find for the point source

$$\rho(\vec{r}) = Qf(\frac{\vec{r}}{r}) \tag{27}$$

and

$$\vec{j}(\vec{r}) = v\frac{\vec{r}}{r} \; \rho(\vec{r}) \tag{28}$$

If in particular

$$f(\vec{\Omega}) = \frac{1}{4\pi} \tag{29}$$

these become:

$$\rho(\vec{r}) = \frac{Q}{4\pi r^2 v} \tag{30}$$

and

$$\vec{j}(\vec{r}) = (\frac{\vec{r}}{r}) \frac{Q}{4\pi r^2} \tag{31}$$

II. Purely Absorbing Medium

5. Consequences of the Continuity Equation

If neutrons can be absorbed by the medium the equation of continuity must be modified by an additional term describing the number of neutrons disappearing per unit time. Denoting the mean life with respect to capture by τ, it is apparent that this number is ψ/τ. The steady state equation of continuity then becomes

$$v\vec{\Omega}\cdot \text{grad}\ \psi(\vec{r},\vec{\Omega}) = q(\vec{r},\vec{\Omega}) - \frac{\psi(\vec{r},\vec{\Omega})}{\tau} \tag{1}$$

It is often convenient to use the mean free path ℓ, and its reciprocal σ, defined by:

$$\ell(\vec{r}) = \frac{1}{\sigma(\vec{r})} = v\,\tau\,(\vec{r}) \tag{2}$$

In terms of σ (1) becomes:

$$\Omega\cdot\text{grad}\ \psi\ (\vec{r},\vec{\Omega}) = \frac{q(\vec{r},\vec{\Omega})}{v} - \sigma\,\psi\,(\vec{r},\vec{\Omega}) \tag{3}$$

Since σ is in general a function of position, (3) is somewhat more complicated than (2-3). However, for σ constant (3) may be multiplied by the integrating factor $e^{(\sigma\vec{r}\cdot\vec{\Omega})}$ becoming:

$$v\,\vec{\Omega}\cdot\text{grad}(\psi\, e^{(\sigma\vec{r}\cdot\vec{\Omega})} = q\ e^{(\sigma\vec{r}\cdot\vec{\Omega})} \tag{4}$$

(4) is then the same as (2-3) with a redefinition of ψ and q. With this in mind the previous results may be applied immediately. Thus for σ constant we find

$$\psi(\vec{r},\vec{\Omega}) = \frac{1}{v}\int_0^\infty q(\vec{r} - R\vec{\Omega},\vec{\Omega})\ e^{-\sigma R}\ dR \tag{5}$$

(5) shows that neutrons coming from the source are attenuated exponentially. For example, if the angular density be everywhere the same on a plane z = 0 and is a function only of $\mu = \vec{\Omega}\cdot\frac{\vec{z}}{z}$ we have

$$n(z,\mu) = n(0,\mu)\ e^{-\sigma\frac{z}{\mu}} \tag{6}$$

Thus even for an isotropic plane source at z = 0 (and hence by (4-10) $n(0,\mu)\sim\frac{1}{|\mu|}$) the angular distribution will become more and more peaked in the forward direction the larger the value of z. This change from the isotropic source distribution is understood by noting that those neutrons which leave z = 0 at an acute angle with this plane take a longer path to reach a distance z from the plane than those travelling at right angles. Hence the former neutrons are more likely to be absorbed.

For problems of plane symmetry (i.e., where the only spatial variation is in terms of a single cartesian coordinate z and the only dependence on $\vec{\Omega}$ is in terms of μ, the cosine of the angle between $\vec{\Omega}$ and the z axis) the occurrence of a variable σ is no difficulty. In these problems (3) reduces to:

$$\mu \frac{\partial \psi}{\partial z} = \frac{q}{v} - \sigma \psi \tag{7}$$

Introducing the new variable y by

$$y = \int_{z_o}^{z} \sigma(z)\, dz \tag{8}$$

where z_o is some conveniently chosen constant, reduces (8) to

$$\mu \frac{\partial \psi}{\partial y} = \frac{\ell(y) q}{v} - \psi \tag{9}$$

(9) then describes a problem with σ = const = 1 with a slightly modified source ($q \rightarrow \ell q$).

In more general problems of variable σ it is convenient to define an optical thickness α by:

$$\alpha(\vec{r}, \vec{r}\,') = \alpha(\vec{r}\,', \vec{r}) = \int_0^R \sigma\left(\vec{r} - s\frac{\vec{R}}{R}\right) ds \tag{10}$$

where $\vec{R} = \vec{r}\,' - \vec{r}$. It is readily shown that, in terms of α, the solution of (3) may be written

$$\psi(\vec{r}, \vec{\Omega}) = \frac{1}{v} \int_0^\infty q(\vec{r} - R\vec{\Omega}, \vec{\Omega})\, e^{-\alpha(\vec{r}, \vec{r} - R\vec{\Omega})} dR \tag{11}$$

and hence that

$$\rho(\vec{r}) = \frac{1}{v} \int \frac{q(\vec{r}',\frac{\vec{R}}{R})}{R^2} \; e^{-\alpha(\vec{r},\vec{r}')} \, d\vec{r}' \tag{12}$$

As is to be expected, this reduces to (5) if σ is constant since then α becomes σR. It should be pointed out, though, that this general case of variable σ introduces considerable complication. Absolute coordinates instead of just relative coordinates between field and source enter the problem.

In the next sections we will apply the solution (11) to various simple sources.

6. Point Source

If a directional point source be located at $\vec{r}_o$ radiating in the direction $\vec{\Omega}_o$

$$q(\vec{r}',\vec{\Omega}) = Q \;\; \delta_2(\vec{\Omega}_o \cdot \vec{\Omega}) \;\; \delta(\vec{r}' - \vec{r}_o) \tag{1}$$

and then

$$\psi(\vec{r},\vec{\Omega}) = \frac{Q}{v|\vec{r} - \vec{r}_o|^2} \;\; \delta_2(\vec{\Omega}_o \cdot \vec{\Omega}) \;\; \delta_2\left(\frac{\vec{r} - \vec{r}_o}{|\vec{r} - \vec{r}_o|} \cdot \Omega\right) \, e^{-\alpha(\vec{r},\vec{r}_o)} \tag{2}$$

For isotropic sources

$$q(\vec{r},\vec{\Omega}) = \frac{1}{4\pi} \, q_o(\vec{r}) \tag{3}$$

and then

$$\psi(\vec{r},\vec{\Omega}) = \frac{1}{4\pi v} \int_0^\infty q_o(\vec{r} - R\vec{\Omega}) \, e^{-\alpha(\vec{r},\vec{r} - R\vec{\Omega})} \, dR \tag{4}$$

$$\rho(\vec{r}) = \frac{1}{4\pi v} \int \frac{q_o(\vec{r}')}{|\vec{r} - \vec{r}'|^2} \;\; e^{-\alpha(\vec{r},\vec{r}')} \, d\vec{r}' \tag{5}$$

If σ be constant and $q_o(\vec{r}) = \delta(\vec{r})$, describing a unit isotropic point source at the origin, (5) becomes

$$\rho(r) = \frac{e^{-\sigma r}}{4\pi r^2 v} \tag{6}$$

7. <u>Plane Source</u>

The neutron distribution arising from an infinite plane ($z = 0$) covered with isotropic point sources of unit strength and density Q_s per unit area can in the case of constant σ be determined directly from (5) or more concisely using the theorem of Appendix B. This states that the plane source solution is connected with the unit point source solution by

$$\rho_{p\ell}(z) = Q_s \int_0^\infty \rho_p(R)\, 2\pi r dr \tag{1}$$

where

$$R^2 = z^2 + r^2$$

Thus, using the point source solution (6), we have

$$\rho_{p\ell}(z) = \frac{Q_s}{4\pi v} \int_0^\infty \frac{e^{-\sigma R}}{R^2}\, 2\pi r dr = \frac{Q_s}{2v} \int_{|z|}^\infty \frac{e^{-\sigma R}}{R}\, dR \tag{2*}$$

$$= \frac{Q_s}{2v} E_1(\sigma |z|)$$

The current due to the plane source is in the + z direction for $z > 0$ and in the minus z direction for $z < 0$. It is of magnitude

$$j = \frac{Q_s}{2} \sigma |z| \int_{\sigma |z|}^\infty \frac{e^{-x}}{x^2}\, dx = \frac{1}{2} Q_s E_2(\sigma |z|) \tag{3}$$

---

*cf. Appendix A where the functions $E_n(x) = \int_1^\infty e^{-xu} u^{-n} du$ are discussed.

Using (2) the neutron density corresponding to an arbitrary distribution of isotropic sources of plane symmetry is obtained immediately. If $q_o(z')\,dz'$ be the number of point sources lying in a region of unit area and thickness dz' at z' the addition of contributions similar to (2) yields

$$\rho(z) = \frac{1}{2v}\int_{-\infty}^{\infty} q_o(z')\, E_1\left(\sigma|z - z'|\right) dz' \tag{4}$$

For σ variable the transformation (5-8) gives

$$\rho(y) = \frac{1}{2v}\int_{-\infty}^{\infty} \ell(y')\, q_o(y')\, E_1(|y - y'|)\, dy' \tag{5}$$

8. <u>Shell Source</u>

By an isotropic shell source of strength Q, radius "a," we mean a sphere of radius "a" uniformly covered with Q isotropic unit point sources. In Appendix B it is shown that the neutron density at r due to a unit isotropic shell source of radius "a" is related to that due to a unit isotropic plane source by

$$\rho_{sh}(r,a) = \frac{1}{4\pi ar}\left\{\rho_{p\ell}(|a - r|) - \rho_{p\ell}(|a + r|)\right\} \tag{1}$$

Hence for a shell source of strength Q, radius r', we have for constant σ

$$\rho_{sh}(r,r') = \frac{Q}{8\pi rr'v}\left\{E_1(\sigma|r - r'|) - E_1(\sigma|r + r'|)\right\} \tag{2}$$

Similarly,

$$j_{sh}(r,r') = \frac{Q}{8\pi rr'}\left\{\frac{E_3(\sigma|r-r'|) - E_3(\sigma|r+r'|)}{\sigma\quad r} + \frac{(r-r')}{|r-r'|}\, E_2(\sigma|r-r'|) - E_2(\sigma|r+r'|)\right\} \tag{2a}$$

From this the density at r due to the isotropic, spatially spherically symmetrical source distribution $q_o(r')$ is immediately seen to be

$$\rho(r) = \frac{1}{v}\int_{\text{all space}} \frac{q_o(r')}{8\pi rr'}\left\{E_1(\sigma|r-r'|) - E_1(\sigma|r+r'|)\right\} dr' \tag{3}$$

or

$$r\rho(r) = \frac{1}{2v}\int_0^\infty r' q_0(r') \left\{E_1(\sigma|r - r'|) - E_1(\sigma|r + r'|)\right\} dr' \qquad (4)$$

If we define $q_0(-r) = q_0(r)$ and $\rho(-r) = \rho(r)$, (3) becomes:

$$r\rho(r) = \frac{1}{2v}\int_{-\infty}^{\infty} r' q_0(r') E_1(\sigma|r - r'|)\, dr' \qquad (5)$$

Comparing (5) and (7-4) we see that the spherical quantities $r\rho(r)$ and $rq_0(r)$ bear the same relation to each other as do the plane quantities $\rho(z)$ and $q_0(z)$.

9. Remarks on the Reciprocity Theorem

If $\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)$ be the angular density at $\vec{r}$ of neutrons moving in direction $\vec{\Omega}$ due to a unit source at $\vec{r}_0$ radiating in the direction $\vec{\Omega}_0$ equation (6-2) states

$$\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0) = \frac{Q}{v|\vec{r}-\vec{r}_0|^2}\,\delta_2(\vec{\Omega}_0\cdot\vec{\Omega})\,\delta_2\!\left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|}\cdot\vec{\Omega}\right) e^{-\alpha(\vec{r},\vec{r}_0)} \qquad (6\text{-}2)$$

Noting the symmetry of $\alpha(\vec{r},\vec{r}_0)$ expressed by (5-10) we see

$$\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0) = \psi(\vec{r}_0,-\vec{\Omega}_0;\vec{r},-\vec{\Omega}) \qquad (1)$$

This reciprocity theorem thus states that the angular density at $\vec{r}$ in direction $\vec{\Omega}$ due to a unit source at $\vec{r}_0$ radiating in direction $\vec{\Omega}_0$ is the same as the angular density at $\vec{r}_0$ direction $-\vec{\Omega}_0$ due to a unit source at $\vec{r}$ radiating in direction $-\vec{\Omega}$. (In Part C it will be shown that this reciprocity theorem holds even if scattering is considered.)

From this physically obvious result we derive on integrating (1) over $\vec{\Omega}$

$$\int \psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\, d\Omega = \int \psi(\vec{r}_0,-\vec{\Omega}_0;\vec{r},-\vec{\Omega})\, d\Omega \qquad (2)$$

The left hand side is the density at $\vec{r}$ due to the directed source at $\vec{r}_o$ while the right hand side is the angular density at $\vec{r}_o$ due to the unit isotropic source at $\vec{r}$.

Integrating (2) over $\vec{\Omega}_o$ yields the additional result that the density at $\vec{r}$ due to the unit isotropic source at $\vec{r}_o$ is equal to the density at $\vec{r}_o$ due to a unit isotropic source at $\vec{r}$.

Multiplying (1) by $\vec{\Omega}$ and integrating over both $\vec{\Omega}_o$ and $\vec{\Omega}$ gives

$$\iint \vec{\Omega}\psi(\vec{r},\vec{\Omega};\ \vec{r}_o,\vec{\Omega}_o)\ d\Omega d\Omega_o = \iint \vec{\Omega}\,\psi(\vec{r}_o,\ -\vec{\Omega}_o;\ \vec{r},\ -\vec{\Omega})\ d\Omega d\Omega_o \tag{3}$$

On taking the scalar product of (3) with an arbitrary unit vector $\vec{n}$ we see this states that the component of the current at $\vec{r}$ in direction $\vec{n}$ due to a unit isotropic source at $\vec{r}_o$ is equal to the density at $\vec{r}_o$ due to a source at $\vec{r}$ with directional distribution $-\frac{1}{4\pi}\ \Omega \cdot n$.

10. Escape Probabilities

As an application of the formulae so far derived we will now turn to the calculation of escape probabilities.

Let a neutron born at a point $\vec{r}$ of a finite absorbing body have a probability $P(\vec{r})$ of escaping from the body. If neutrons are born within the body with isotropic distribution $q_o(\vec{r})$ the <u>average escape probability</u> (P_o) is:

$$P_o = \frac{\int_{body} q_o(\vec{r})\ P(\vec{r})\ d\vec{r}}{\int_{body} q_o(\vec{r})\ d\vec{r}} \tag{1}$$

The average probability P_c ($P_c = 1 - P_o$) that the neutron will be absorbed is often referred to as the collision probability.

In the following we shall treat the case of constant q_o. Then we have

$$P_o = \frac{1}{V} \int P(\vec{r})\, dV \qquad (2)^*$$

As a simple illustration we calculate the escape probability from a thin slab of thickness "a" (Fig. 6)

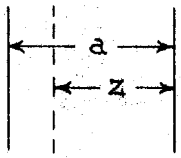

Fig. 6

For a neutron starting at z the escape probability is equal to the sum of the current on the two sides of the slab due to a unit source at z. Since these sides are at distances z and a-z from the origin of the neutron we have from (8-3)

$$P(z) = \frac{1}{2} \left\{ E_2\left(\frac{z}{\ell}\right) + E_2\left(\frac{a-z}{\ell}\right) \right\} \qquad (3)$$

With $a/\ell << 1$ (and hence $z/\ell << 1$) it is reasonable to use the expansion

$$E_2(x) = 1 + x \log x - 0.422784\, x + O(x^2) \qquad (4)$$

giving

$$1 - P\left(\frac{z}{\ell}\right) = \frac{a}{2} \log \frac{\ell}{a} + \frac{a}{\ell} f\left(\frac{z}{a}\right) \qquad (5)$$

where

$$f(x) = \frac{1}{2} \left[0.422784 - x \log x - (1 - x) \log (1 - x) \right] \qquad (6)$$

$f(x)$ varies between $f(0) = f(1) = 0.211$ and $f(\frac{1}{2}) = 0.211 + \frac{1}{2} \log 2 = 0.558$. Thus for a very thin slab ($\log \frac{\ell}{a} >> 1$) the escape probability from a point z becomes independent of z and equal to the average escape probability, i.e.

$$1 - P_o \cong \frac{a}{2\ell} \log \frac{\ell}{a} \qquad (7)$$

Physically the constancy of P(z) may be understood as follows: A neutron

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*This P_o has been used for approximate treatment of pile problems and for the determination of critical sizes. If c neutrons be emitted following each absorption the critical condition is given by $cP_c = c(1 - P_o) = 1$.

starting at the center of a thin slab need traverse a distance of only $\frac{a}{2}$ to leave the body. It sees material of thickness $\frac{a}{2}$ on both sides. A neutron originating at the left edge escapes if it moves to the left. However, if it moves to the right it must traverse the entire distance "a". These two factors tend to equalize and for a thin slab center and edge escape probabilities become approximately equal.

For larger "a" the escape probability is no longer position independent and P_o must be evaluated explicitly.

$$P_o(a) = \frac{1}{a} \int_0^a P(z)\, dz = \frac{\ell}{a} \int_0^{a/\ell} E_2(z)\, dz \tag{8}$$

$$= \frac{\ell}{a} \left\{ \frac{1}{2} - E_3\left(\frac{a}{\ell}\right) \right\}$$

For $a/\ell >> 1$ (8) becomes

$$P_o(a) \approx \frac{\ell}{2a} - \frac{e^{-a/\ell}}{(a/\ell)^2} \tag{9}$$

In general for convex bodies $P(\vec{r})$ is the total current emerging from the surface of a body due to a unit isotropic source at point $\vec{r}$. With the notation of Fig. 7 this gives

$$P(\vec{r}) = \frac{1}{4\pi} \int \left(\frac{\vec{R}}{R} \cdot \vec{n}\right) \frac{e^{-R/\ell}}{R^2}\, dS \tag{10}$$

where dS denotes the surface element.

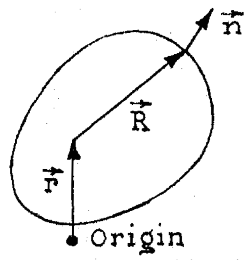

Fig. 7

If $d\Omega$ be the solid angle subtended by dS at $\vec{r}$

$$d\Omega = \frac{\vec{R}\cdot n}{R} \frac{dS}{R^2} \tag{11}$$

and hence

$$P(\vec{r}) = \frac{1}{4\pi} \int e^{-R/\ell} \, d\Omega \tag{12}$$

which gives on combining with (2)

$$P_o = \frac{1}{4\pi V} \iint e^{-R/\ell} \, d\Omega dV \tag{13}$$

(13) is quite obvious since the probability the neutron will be emitted in Ω is, by assumption, $\frac{d\Omega}{4\pi}$. The probability that, if emitted in direction $\vec{\Omega}$, it escapes is $e^{-R/\ell}$. The integral over V and division by it is just averaging over all points of the volume. Moreover, in this form the generalization to non-convex bodies is apparent. Formula (13) remains valid with the understanding that R is the total thickness of the body in direction $\vec{\Omega}$ from $\vec{r}$.

Equation (13) for the escape probability may be re-expressed by interchanging the order of integration. Thinking of the volume V as being made up of tubes having a cross-section $(\vec{\Omega}\cdot\vec{n}_i)$ dS (where $\vec{n}_i = -\vec{n}$ is the inner normal at dS) and a length R_s, where R_s is the length of the chord drawn in direction $\vec{\Omega}$ from the surface element dS (Fig. 8) we see

$$dV = (\vec{\Omega}\cdot\vec{n}_i) \, dS \, dR \tag{14}$$

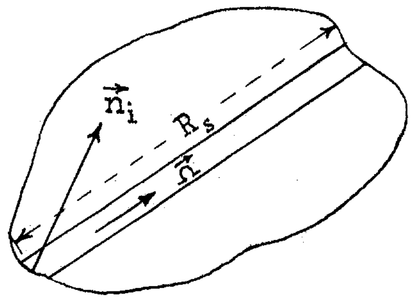

Fig. 8

Hence

$$P_o = \frac{1}{4\pi V} \iiint_0^{R_s} e^{-R/\ell} \, dR \, (\vec{\Omega}\cdot\vec{n}_i) \, d\Omega dS \tag{15}$$

where the integration region is such that $\vec{\Omega}\cdot\vec{n}_i > 0$.

Carrying out the R integration yields

$$P_o = \frac{\ell}{4\pi V} \iint (1 - e^{-R_s/\ell}) (\vec{\Omega}\cdot\vec{n}_1)\, d\Omega\, dS \tag{16}$$

An alternate derivation of (16) is obtained by relating the escape probability to the solution of another problem. This may be done using the reciprocity theorem.

Consider the body for which the escape probability is required to be immersed in a uniform isotropic bath of neutrons. Let the (constant) neutron density at large distances from the body be ρ_∞. The number of neutrons entering the body per second through the surface element dS direction $\vec{\Omega}_o$ is then

$$\frac{\rho_\infty V}{4\pi} \vec{\Omega}_o\cdot\vec{n}\, dS$$

where $\vec{n}$ is the normal at dS. If as before $\psi(\vec{r},\vec{\Omega};\ \vec{r}_o,\vec{\Omega}_o)$ is the angular density at $\vec{r}$ in direction $\vec{\Omega}$ due to a unit source at $\vec{r}_o$ radiating in the direction $\vec{\Omega}_o$ we have for a point $\vec{r}$ in the body

$$\begin{aligned}\rho(\vec{r}) &= \int \psi(\vec{r},\vec{\Omega})\, d\Omega \\ &= \frac{\rho_\infty V}{4\pi} \iiint \vec{\Omega}_o\cdot\vec{n}\, \psi(\vec{r},\vec{\Omega};\ \vec{r}_o,\vec{\Omega}_o)\, d\Omega_o\, d\Omega\, dS\end{aligned} \tag{17}$$

By the reciprocity theorem

$$\psi(\vec{r},\vec{\Omega};\ \vec{r}_o,\vec{\Omega}_o) = \psi(\vec{r}_o,\ -\vec{\Omega}_o;\ \vec{r},\ -\vec{\Omega}) \tag{18}$$

Inserting (17) into (18):

$$\rho(\vec{r}) = \rho_\infty \left\{ \frac{V}{4\pi} \iiint \vec{\Omega}_o\cdot\vec{n}\, \psi(\vec{r}_o,\ -\vec{\Omega}_o;\ \vec{r},\ -\vec{\Omega})\, d\Omega_o\, d\Omega\, dS \right\} \tag{19}$$

On changing the signs of the variables of integration ($\vec{\Omega}_o$ and $\vec{\Omega}$) it is seen that the term in brackets is just $P(\vec{r})$. Hence

$$P(\vec{r}) = \frac{\rho(\vec{r})}{\rho_\infty} \tag{20}$$

Integrating over the body and dividing by the volume yields

$$P_o = \frac{\rho_{av}}{\rho_\infty} \tag{21}$$

where ρ_{av} is the average neutron density in the body. Thus the escape probability problem is equivalent to asking for the neutron density in a body immersed in an isotropic bath.

Let N_i be the number of neutrons entering the body from the bath per second. We have

$$N_i = \iint \frac{v}{4\pi} \rho_\infty \vec{\Omega} \cdot \vec{n}_i \, d\Omega dS \tag{22}$$

since $\frac{v\vec{\Omega}\rho_\infty}{4\pi}$ is the incident angular current.

The number of neutrons leaving each second (N_o) is

$$N_o = \iint e^{-R_s/\ell} \frac{v}{4\pi} \rho_\infty \vec{\Omega} \cdot \vec{n}_i \, d\Omega dS \tag{23}$$

as a beam entering in a direction $\vec{\Omega}$ at dS is attenuated by $e^{-R_s/\ell}$ before leaving. The number of neutrons absorbed per second (N_a) is

$$N_a = N_i - N_o = \frac{\rho_\infty v}{4\pi} \iint (1 - e^{-R_s/\ell}) \vec{\Omega} \cdot \vec{n}_i \, d\Omega dS \tag{24}$$

But

$$N_a = \frac{v}{\ell} V \rho_{av} \tag{25}$$

Equating (24) and (25) gives

$$P_o = \frac{\rho_{av}}{\rho_{as}} = \frac{\ell}{4\pi V} \iint (1 - e^{-R_s/\ell}) \vec{\Omega} \cdot \vec{n}_i \, d\Omega dS \tag{26}$$

On performing the integration in (22) we find

$$N_i = \frac{v \rho_\infty S}{4V} \tag{27}$$

Therefore if F be the fraction of incoming neutrons absorbed we see

$$F = \frac{N_a}{N_i} = \frac{4V}{4\pi S} \iint (1 - e^{-R_s/\ell}) \; \vec{\Omega}\cdot\vec{n}_i \; d\Omega dS \qquad (28)$$

giving the alternative expression for P_o

$$P_o = \frac{\ell S}{4V} \; F \qquad (29)$$

It may be noted that the consideration of this equivalent neutron bath problem is somewhat more than an alternative means of computation. Using (29), optical analogy experiments with light taking the place of neutrons have been performed to determine escape probabilities experimentally.

10.1 The Chord Method

The calculation of escape probabilities is often facilitated by the chord method developed by Dirac*. Though the limitation is not essential, we will only consider convex bodies in our discussion.

Fig. 9 shows chords of varying length R_s drawn from the surface element dS of a convex body.

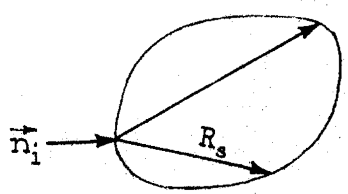

Fig. 9

Let such chords be drawn so that their number in a given direction is proportional to $\vec{\Omega}\cdot\vec{n}_i$; the cosine of the angle between the inner surface normal and the chord direction. Defining $\phi(R)dR$ as the probability that a chord be of length between R and R + dR yields

*P.A.M. Dirac, Approximate Rate of Neutron Multiplication for a Solid of Arbitrary Shape and Uniform Density, British Declassified Document MS.D.5 Part 1, 1943.

$$\phi(R)dR = \frac{\iint\limits_{R_s = R} (\vec{\Omega}\cdot\vec{n}_1)\, d\Omega\, dS}{\iint \vec{\Omega}\cdot\vec{n}_1\, d\Omega\, dS} \tag{1}$$

where the only restriction on the region of integration in the denominator is $\vec{\Omega}\cdot\vec{n}_1 > 0$.

(1) is derived as follows: If $\vec{\Omega}$ be a direction from a surface element dS such that $R_s = R$ and $d\Omega$ be a solid angle around $\vec{\Omega}$ such that all chords in this solid angle are within dR of length R, the number of chords of lengths R to R + dR from this solid angle are by assumption proportional $\vec{\Omega}\cdot\vec{n}_1 d\Omega\, dS$. Summing over all directions $\vec{\Omega}$ of this kind and then over all points on the surface gives the total number of these chords as proportional to the numerator of (1). Dividing by the total number of chords drawn (which then is proportional to the denominator of (1) with the same proportionality constant) shows that (1) is indeed the requisite probability.

Integrating the denominator of (1) gives

$$\int (\vec{\Omega}\cdot\vec{n}_1)\, d\Omega\, dS = 2\pi S \int_0^1 \mu\, d\mu = \pi S \tag{2}$$

where S signifies the total surface area. The <u>average chord length</u> R_{av} is

$$R_{av} = \int R\, \phi(R)\, dR$$

$$= \frac{\int R\, \vec{\Omega}\cdot\vec{n}_1 d\Omega\, dS}{\pi S} \tag{3}$$

Since

$$\int R\, \vec{\Omega}\cdot\vec{n}_1 dS = V \tag{4}$$

we have

$$R_{av} = \frac{4V}{S} \tag{5}$$

and thus

$$\frac{4\pi V}{R_{av}} \phi(R)\, dR = \int_{R_s = R} \vec{\Omega}\cdot\vec{n}_1\, d\Omega\, dS \qquad (6)$$

On inserting (6) into (10-16) we find

$$P_o = \frac{\ell}{R_{av}} \int (1 - e^{-R/\ell})\, \phi(R)\, dR \qquad (7)$$

The determination of P_o is thus divided into the geometrical problem of finding $\phi(R)$ and the integration indicated in (7).

Useful limiting forms for P_o are obtainable from (7) with but little work. Thus, if the body be of such small dimensions that for all R $R \ll \ell$, the expression for P_o may be simplified by expanding the exponential in (7) giving

$$P_o \approx \frac{\ell}{R_{av}} \int \left(\frac{R}{\ell} - \frac{1}{2}\frac{R^2}{\ell^2}\right) \phi(R)\, dR \qquad (8)$$

or

$$P_o \approx 1 - \frac{1}{2}\frac{(R^2)_{av}}{\ell R_{av}} \qquad (9)$$

For bodies of dimensions large compared with the mean free path it is convenient to write (7) as

$$P_o = \frac{\ell}{R_{av}} \left\{ 1 - \int e^{-R/\ell} \phi(R)\, dR \right\} \qquad (10)$$

Since in this case only small values of R will contribute appreciably (due to the exponential factor) it is reasonable to expand

$$\phi(R) = \phi(0) + R\phi'(0) + \ldots \qquad (11)$$

Inserting in (10) gives:

$$P_o \approx \frac{\ell}{R_{av}} \left\{ 1 - \ell\phi(0) - \ell^2\ \phi'(0) \right\} \qquad (12)$$

If "a" be a typical dimension of the body (12) can be written

$$P_o \approx \frac{\ell}{R_{av}} \left\{ 1 - \frac{\ell}{a} c_1 - \left(\frac{\ell}{a}\right)^2 \frac{c_2}{2} \right\} \tag{13}$$

where c_1 and c_2 are some constants.

For smooth bodies (i.e., no edges) one can show that $\phi(0)$ vanishes. Hence for these

$$P_o \approx \frac{S\ell}{4V} \left\{ 1 - 0\left[\left(\frac{\ell}{a}\right)^2\right] \right\} \tag{14}$$

For bodies with edges this is no longer true and

$$P_o \approx \frac{S\ell}{4V} \left\{ 1 - 0\left(\frac{\ell}{a}\right) \right\} \tag{15}$$

For bodies whose dimensions are comparable with the mean free path much more detailed calculations are necessary. In the following exact escape probabilities for bodies of various special shapes will be found using the chord method.

10.2 Slab and Sphere

Referring to Fig. 10 we note that for a slab of thickness "a"

$$d\Omega = d(\cos\theta)\, d\phi = d\mu\, d\phi \tag{1}$$

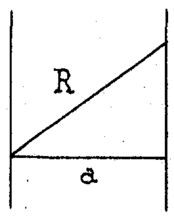

Fig. 10

Therefore

$$\phi(R)dR = \frac{1}{\pi S} \int_{R_s = R} (\vec{\Omega} \cdot \vec{n}_1)\, d\mu\, d\phi\, dS \tag{2}$$

$$= 2\mu\, d\mu$$

Since

$$\mu = \frac{a}{R} \qquad (3)$$

$$\phi(R)dR = 2a^2 \frac{dR}{R^3} \qquad (4)$$

Using

$$\frac{4V}{S} = 2a \qquad (5)$$

We find by (10.1-7)

$$P_o = \ell a \int_a^\infty (1 - e^{-R/\ell}) \frac{dR}{R^3} = \frac{\ell}{a} \left\{ \frac{1}{2} - E_3 \left(\frac{a}{\ell}\right) \right\} \qquad (6)$$

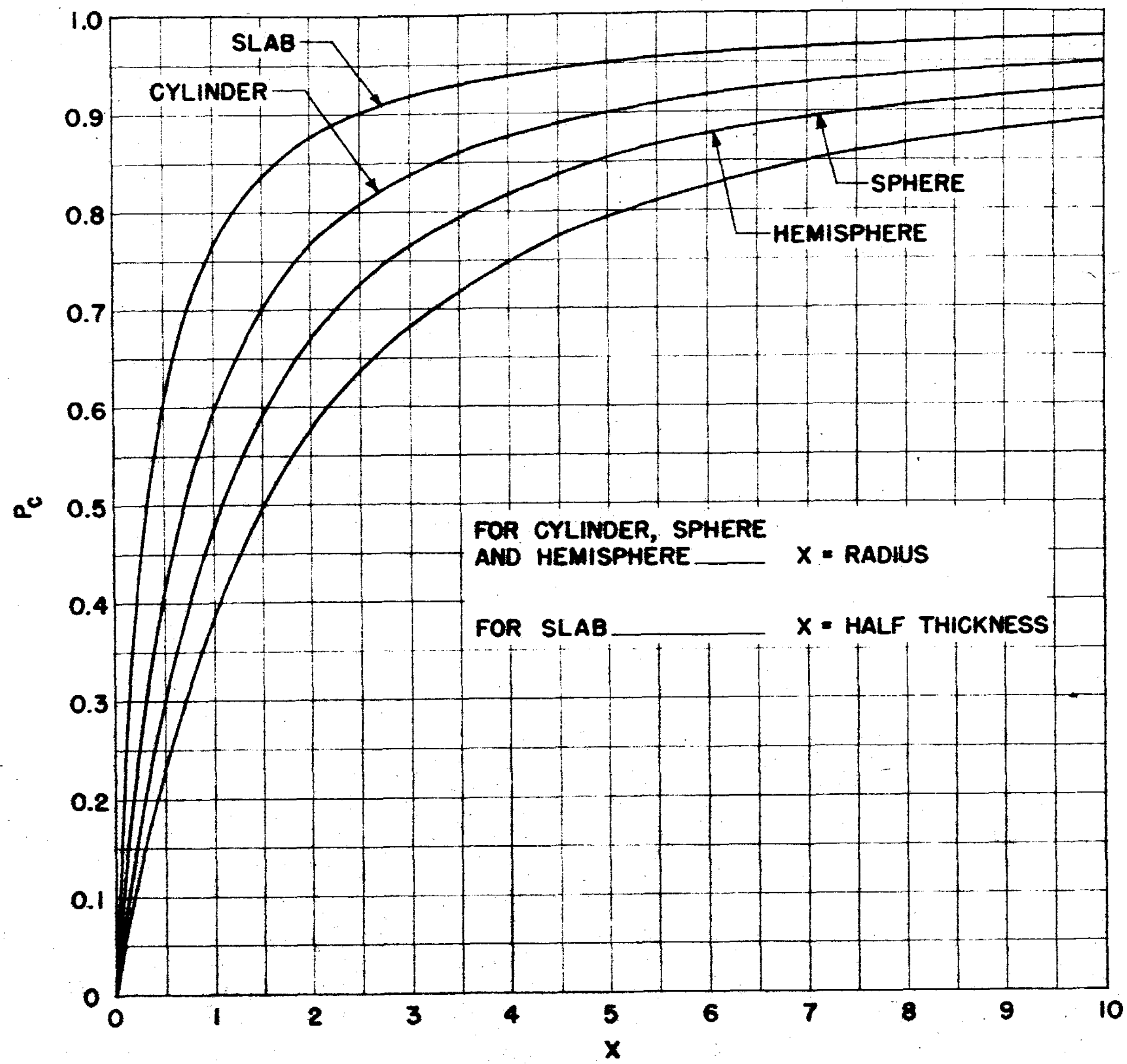

Fig. 11
COLLISION PROBABILITIES FOR UNIFORM SOURCE DENSITY

(6) is, of course, identical with the previously derived result (10-8). For $a/\ell >> 1$ we found

$$P_o \approx \ell/2a \qquad a/\ell >> 1 \tag{7}$$

If $a/\ell << 1$ expanding E_3 gives

$$P_o = 1 - \frac{a/\ell}{2} \log \frac{1}{a/\ell} - \frac{a/\ell}{2}\left[1.5 - \gamma\right] - \frac{1}{6}(a/\ell)^2 \tag{8}$$

where γ = Euler's Constant = 0.577216 ...

Collision probabilities calculated from (6) are given in Table 2 as a function of b/ℓ with $b = a/2$ = half thickness of the slab. A graph of escape probability vs. half thickness is given in Fig. 11.

For a sphere of radius "a" (Fig. 12) it is convenient to introduce as a variable θ_o, the angle between the chord and the radius vector.

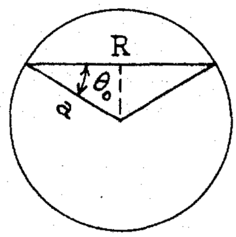

Fig. 12

Then

$$\cos\theta_o = \mu_o = \frac{R}{2a} \tag{9}$$

This gives:

$$\phi(R)dR = \frac{2\pi\mu_o d\mu_o}{\pi} = \frac{R}{2a^2}\, dR \tag{10}$$

also

$$R_{av} = \frac{4V}{S} = \frac{4}{3}a \tag{11}$$

Hence:

$$P_o = \frac{3\ell}{4a}\int_0^{2a}(1 - e^{-R/\ell})\, RdR/2a^2 \tag{12}$$

TABLE 2

TABLE OF P_c FOR SLAB

Formula:

$$P_c = 1 - \frac{\ell}{2b}\left\{\frac{1}{2} - E_3\left(\frac{2b}{\ell}\right)\right\}$$

Expansions: (4-decimal accuracy)

(1) $$P_c = \frac{b}{\ell}\left[\frac{3}{2} - \gamma + \log\frac{\ell}{2b}\right] + \frac{2}{3}(b/\ell)^2 + \ldots \quad 0 \leq \frac{2b}{\ell} < 0.20$$

(2) $$P_c = 1 - \frac{\ell}{4b} \quad \text{for} \quad 3 < \frac{b}{\ell} < \infty$$

$$\gamma = 0.577216$$

b/ℓ	P_c	b/ℓ	P_c	b/ℓ	P_c	b/ℓ	P_c
.00	.0000	.25	.4432	.50	.6097	.75	.7045
.01	.0484	.26	.4523	.51	.6145	.76	.7074
.02	.0831	.27	.4611	.52	.6192	.77	.7103
.03	.1127	.28	.4696	.53	.6237	.78	.7132
.04	.1390	.29	.4779	.54	.6282	.79	.7159
.05	.1629	.30	.4859	.55	.6326	.80	.7187
.06	.1849	.31	.4937	.56	.6369	.81	.7214
.07	.2054	.32	.5013	.57	.6411	.82	.7240
.08	.2246	.33	.5087	.58	.6453	.83	.7266
.09	.2427	.34	.5159	.59	.6493	.84	.7292
.10	.2597	.35	.5229	.60	.6533	.85	.7317
.11	.2759	.36	.5298	.61	.6572	.86	.7342
.12	.2913	.37	.5364	.62	.6610	.87	.7366
.13	.3060	.38	.5429	.63	.6647	.88	.7390
.14	.3200	.39	.5492	.64	.6684	.89	.7414
.15	.3335	.40	.5554	.65	.6720	.90	.7437
.16	.3464	.41	.5614	.66	.6755	.91	.7460
.17	.3588	.42	.5673	.67	.6790	.92	.7483
.18	.3707	.43	.5730	.68	.6824	.93	.7505
.19	.3821	.44	.5786	.69	.6857	.94	.7527
.20	.3932	.45	.5841	.70	.6890	.95	.7548
.21	.4039	.46	.5895	.71	.6922	.96	.7569
.22	.4142	.47	.5947	.72	.6954	.97	.7590
.23	.4242	.48	.5998	.73	.6985	.98	.7611
.24	.4339	.49	.6048	.74	.7015	.99	.7631
.25	.4432	.50	.6097	.75	.7045	1.00	.7651

b/ℓ	P_c	b/ℓ	P_c	b/ℓ	P_c	b/ℓ	P_c
1.00	.7651	1.50	.8363	2.00	.8757	2.50	.9002
1.01	.7670	1.51	.8373	2.01	.8763	2.51	.9006
1.02	.7690	1.52	.8383	2.02	.8769	2.52	.9010
1.03	.7709	1.53	.8393	2.03	.8775	2.53	.9013
1.04	.7727	1.54	.8403	2.04	.8781	2.54	.9017
1.05	.7746	1.55	.8413	2.05	.8786	2.55	.9021
1.06	.7764	1.56	.8422	2.06	.8792	2.56	.9025
1.07	.7782	1.57	.8432	2.07	.8798	2.57	.9029
1.08	.7800	1.58	.8441	2.08	.8804	2.58	.9032
1.09	.7817	1.59	.8450	2.09	.8809	2.59	.9036
1.10	.7834	1.60	.8460	2.10	.8815	2.60	.9040
1.11	.7851	1.61	.8469	2.11	.8820	2.61	.9043
1.12	.7868	1.62	.8478	2.12	.8826	2.62	.9047
1.13	.7884	1.63	.8486	2.13	.8831	2.63	.9051
1.14	.7901	1.64	.8495	2.14	.8836	2.64	.9054
1.15	.7917	1.65	.8504	2.15	.8842	2.65	.9058
1.16	.7932	1.66	.8512	2.16	.8847	2.66	.9061
1.17	.7948	1.67	.8521	2.17	.8853	2.67	.9065
1.18	.7963	1.68	.8529	2.18	.8857	2.68	.9068
1.19	.7978	1.69	.8538	2.19	.8863	2.69	.9072
1.20	.7993	1.70	.8546	2.20	.8868	2.70	.9075
1.21	.8008	1.71	.8554	2.21	.8873	2.71	.9078
1.22	.8023	1.72	.8562	2.22	.8878	2.72	.9082
1.23	.8037	1.73	.8570	2.23	.8883	2.73	.9085
1.24	.8051	1.74	.8578	2.24	.8887	2.74	.9089
1.25	.8065	1.75	.8586	2.25	.8892	2.75	.9092
1.26	.8079	1.76	.8593	2.26	.8897	2.76	.9095
1.27	.8093	1.77	.8601	2.27	.8902	2.77	.9098
1.28	.8106	1.78	.8608	2.28	.8907	2.78	.9102
1.29	.8119	1.79	.8616	2.29	.8911	2.79	.9105
1.30	.8132	1.80	.8623	2.30	.8916	2.80	.9108
1.31	.8145	1.81	.8631	2.31	.8921	2.81	.9111
1.32	.8158	1.82	.8638	2.32	.8925	2.82	.9114
1.33	.8171	1.83	.8645	2.33	.8930	2.83	.9117
1.34	.8183	1.84	.8652	2.34	.8934	2.84	.9120
1.35	.8196	1.85	.8659	2.35	.8939	2.85	.9124
1.36	.8208	1.86	.8666	2.36	.8943	2.86	.9127
1.37	.8220	1.87	.8673	2.37	.8948	2.87	.9130
1.38	.8232	1.88	.8680	2.38	.8952	2.88	.9133
1.39	.8243	1.89	.8687	2.39	.8956	2.89	.9136
1.40	.8255	1.90	.8693	2.40	.8961	2.90	.9139
1.41	.8266	1.91	.8700	2.41	.8965	2.91	.9141
1.42	.8278	1.92	.8707	2.42	.8969	2.92	.9144
1.43	.8289	1.93	.8713	2.43	.8973	2.93	.9147
1.44	.8300	1.94	.8720	2.44	.8977	2.94	.9150
1.45	.8311	1.95	.8726	2.45	.8982	2.95	.9153
1.46	.8321	1.96	.8732	2.46	.8986	2.96	.9156
1.47	.8332	1.97	.8738	2.47	.8990	2.97	.9159
1.48	.8342	1.98	.8745	2.48	.8994	2.98	.9162
1.49	.8353	1.99	.8751	2.49	.8998	2.99	.9164
1.50	.8363	2.00	.8757	2.50	.9002	3.00	.9167

b/ℓ	P_c	b/ℓ	P_c	b/ℓ	P_c	b/ℓ	P_c
3.00	.9167	3.50	.9286	4.00	.9375	4.50	.9444
3.01	.9170	3.51	.9288	4.01	.9377	4.51	.9446
3.02	.9173	3.52	.9290	4.02	.9378	4.52	.9447
3.03	.9175	3.53	.9292	4.03	.9380	4.53	.9448
3.04	.9178	3.54	.9294	4.04	.9381	4.54	.9449
3.05	.9181	3.55	.9296	4.05	.9383	4.55	.9451
3.06	.9183	3.56	.9298	4.06	.9384	4.56	.9452
3.07	.9186	3.57	.9300	4.07	.9386	4.57	.9453
3.08	.9189	3.58	.9302	4.08	.9387	4.58	.9454
3.09	.9191	3.59	.9304	4.09	.9389	4.59	.9455
3.10	.9194	3.60	.9306	4.10	.9390	4.60	.9457
3.11	.9196	3.61	.9308	4.11	.9392	4.61	.9458
3.12	.9199	3.62	.9309	4.12	.9393	4.62	.9459
3.13	.9202	3.63	.9311	4.13	.9395	4.63	.9460
3.14	.9204	3.64	.9313	4.14	.9396	4.64	.9461
3.15	.9207	3.65	.9315	4.15	.9398	4.65	.9462
3.16	.9209	3.66	.9317	4.16	.9399	4.66	.9464
3.17	.9212	3.67	.9319	4.17	.9400	4.67	.9465
3.18	.9214	3.68	.9321	4.18	.9402	4.68	.9466
3.19	.9217	3.69	.9323	4.19	.9403	4.69	.9467
3.20	.9219	3.70	.9324	4.20	.9405	4.70	.9468
3.21	.9221	3.71	.9326	4.21	.9406	4.71	.9469
3.22	.9224	3.72	.9328	4.22	.9408	4.72	.9470
3.23	.9226	3.73	.9330	4.23	.9409	4.73	.9471
3.24	.9229	3.74	.9332	4.24	.9410	4.74	.9473
3.25	.9231	3.75	.9333	4.25	.9412	4.75	.9474
3.26	.9233	3.76	.9335	4.26	.9413	4.76	.9475
3.27	.9236	3.77	.9337	4.27	.9415	4.77	.9476
3.28	.9238	3.78	.9339	4.28	.9416	4.78	.9477
3.29	.9240	3.79	.9340	4.29	.9417	4.79	.9478
3.30	.9243	3.80	.9342	4.30	.9419	4.80	.9479
3.31	.9245	3.81	.9344	4.31	.9420	4.81	.9480
3.32	.9247	3.82	.9346	4.32	.9421	4.82	.9481
3.33	.9249	3.83	.9347	4.33	.9423	4.83	.9482
3.34	.9252	3.84	.9349	4.34	.9424	4.84	.9483
3.35	.9254	3.85	.9351	4.35	.9425	4.85	.9485
3.36	.9256	3.86	.9352	4.36	.9427	4.86	.9486
3.37	.9258	3.87	.9354	4.37	.9428	4.87	.9487
3.38	.9261	3.88	.9356	4.38	.9429	4.88	.9488
3.39	.9263	3.89	.9357	4.39	.9431	4.89	.9489
3.40	.9265	3.90	.9359	4.40	.9432	4.90	.9490
3.41	.9267	3.91	.9361	4.41	.9433	4.91	.9491
3.42	.9269	3.92	.9362	4.42	.9434	4.92	.9492
3.43	.9271	3.93	.9364	4.43	.9436	4.93	.9493
3.44	.9273	3.94	.9366	4.44	.9437	4.94	.9494
3.45	.9276	3.95	.9367	4.45	.9438	4.95	.9495
3.46	.9278	3.96	.9369	4.46	.9439	4.96	.9496
3.47	.9280	3.97	.9370	4.47	.9441	4.97	.9497
3.48	.9282	3.98	.9372	4.48	.9442	4.98	.9498
3.49	.9284	3.99	.9373	4.49	.9443	4.99	.9499
3.50	.9286	4.00	.9375	4.50	.9444	5.00	.9500

On integration this gives:

$$P_o = \frac{3}{8(a/\ell)^3} \left\{ 2\,(a/\ell)^2 - 1 + (1 + 2a/\ell)\, e^{-2a/\ell} \right\} \qquad (13)$$

In case $a \ll \ell$ we apply (10.1-9) for which purpose we calculate $(R^2)_{av}$.

$$(R^2)_{av} = \frac{1}{2a^2} \int_0^{2a} R^3 dR = 2a^2 \qquad (14)$$

Hence:

$$P_o \approx 1 - \frac{1}{2}\, \frac{2a^2}{\ell(\frac{4}{3})a} = 1 - \frac{3a}{4\ell} \qquad a \ll \ell \qquad (15)$$

For $a \gg \ell$ expanding (13) gives

$$P_o \approx \frac{3}{4}\, \frac{\ell}{a} \left\{ 1 - \frac{1}{2} \left(\frac{\ell}{a}\right)^2 \right\} \qquad (16)$$

in agreement with (10.1-11).

Table 3 gives the collision probability P_c as a function of a/ℓ. In Fig. 11 there is a graph of P_c vs a/ℓ.

10.3 The Infinite Cylinder

For an infinite cylinder we take coordinates as shown in Fig. 13.

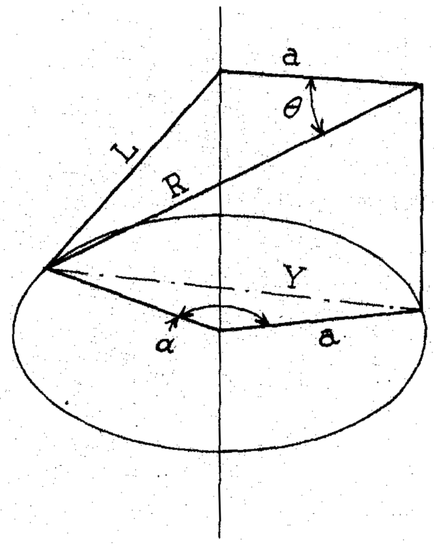

Fig. 13

Now

$$d\Omega = \frac{\mu a\, d\alpha\, dz}{R^2} \qquad (1)$$

TABLE 3

TABLE OF P_c FOR SPHERICAL GEOMETRY

Formula:

$$P_c = 1 - \frac{3}{8\left(\frac{a}{\ell}\right)^3} \left\{ 2\left(\frac{a}{\ell}\right)^2 - 1 + \left(1 + 2\frac{a}{\ell}\right) e^{-2a/\ell} \right\}$$

Expansions: (5-decimal accuracy)

$$(1) \quad P_c = \frac{3}{8}\left(\frac{2a}{\ell}\right) - \frac{1}{10}\left(\frac{2a}{\ell}\right)^2 + \frac{1}{48}\left(\frac{2a}{\ell}\right)^3 + \ldots \quad 0 \leq \frac{a}{\ell} < 0.10$$

$$(2) \quad P_c = 1 - \frac{3\ell}{4a} + \frac{3}{\left(2\frac{a}{\ell}\right)^3}, \quad 4.50 < \frac{a}{\ell} < \infty$$

a/ℓ	P_c	a/ℓ	P_c	a/ℓ	P_c	a/ℓ	P_c
.00	.00000	.25	.16490	.50	.29272	.75	.39304
.01	.00746	.26	.17065	.51	.29722	.76	.39659
.02	.01484	.27	.17634	.52	.30165	.77	.40010
.03	.02214	.28	.18197	.53	.30606	.78	.40358
.04	.02937	.29	.18755	.54	.31041	.79	.40703
.05	.03652	.30	.19308	.55	.31473	.80	.41044
.06	.04360	.31	.19854	.56	.31902	.81	.41384
.07	.05060	.32	.20395	.57	.32325	.82	.41719
.08	.05752	.33	.20931	.58	.32744	.83	.42052
.09	.06438	.34	.21462	.59	.33160	.84	.42382
.10	.07116	.35	.21987	.60	.33573	.85	.42708
.11	.07787	.36	.22507	.61	.33980	.86	.43032
.12	.08452	.37	.23022	.62	.34384	.87	.43353
.13	.09109	.38	.23531	.63	.34783	.88	.43671
.14	.09760	.39	.24036	.64	.35179	.89	.43986
.15	.10403	.40	.24537	.65	.35573	.90	.44298
.16	.11041	.41	.25034	.66	.35962	.91	.44608
.17	.11672	.42	.25521	.67	.36347	.92	.44915
.18	.12296	.43	.26007	.68	.36729	.93	.45219
.19	.12914	.44	.26488	.69	.37107	.94	.45521
.20	.13525	.45	.26962	.70	.37481	.95	.45819
.21	.14130	.46	.27432	.71	.37853	.96	.46116
.22	.14729	.47	.27899	.72	.38221	.97	.46409
.23	.15322	.48	.28362	.73	.38586	.98	.46700
.24	.15909	.49	.28821	.74	.38947	.99	.46989
.25	.16490	.50	.29272	.75	.39304	1.00	.47274

a/ℓ	P_c	a/ℓ	P_c	a/ℓ	P_c	a/ℓ	P_c
1.00	.47274	1.50	.58898	2.00	.66758	2.50	.72303
1.01	.47558	1.51	.59086	2.01	.66888	2.51	.72397
1.02	.47839	1.52	.59273	2.02	.67017	2.52	.72490
1.03	.48118	1.53	.59457	2.03	.67146	2.53	.72582
1.04	.48394	1.54	.59641	2.04	.67273	2.54	.72674
1.05	.48668	1.55	.59823	2.05	.67400	2.55	.72766
1.06	.48940	1.56	.60004	2.06	.67525	2.56	.72857
1.07	.49209	1.57	.60183	2.07	.67650	2.57	.72947
1.08	.49476	1.58	.60361	2.08	.67774	2.58	.73037
1.09	.49740	1.59	.60538	2.09	.67897	2.59	.73126
1.10	.50003	1.60	.60713	2.10	.68019	2.60	.73214
1.11	.50263	1.61	.60887	2.11	.68141	2.61	.73303
1.12	.50521	1.62	.61059	2.12	.68261	2.62	.73390
1.13	.50777	1.63	.61231	2.13	.68381	2.63	.73477
1.14	.51030	1.64	.61401	2.14	.68500	2.64	.73564
1.15	.51282	1.65	.61569	2.15	.68618	2.65	.73650
1.16	.51531	1.66	.61737	2.16	.68736	2.66	.73735
1.17	.51778	1.67	.61903	2.17	.68852	2.67	.73820
1.18	.52024	1.68	.62068	2.18	.68968	2.68	.73905
1.19	.52267	1.69	.62232	2.19	.69083	2.69	.73989
1.20	.52508	1.70	.62394	2.20	.69197	2.70	.74072
1.21	.52747	1.71	.62556	2.21	.69311	2.71	.74155
1.22	.52984	1.72	.62716	2.22	.69424	2.72	.74238
1.23	.53219	1.73	.62875	2.23	.69536	2.73	.74320
1.24	.53452	1.74	.63033	2.24	.69647	2.74	.74402
1.25	.53684	1.75	.63189	2.25	.69758	2.75	.74483
1.26	.53913	1.76	.63345	2.26	.69868	2.76	.74563
1.27	.54141	1.77	.63499	2.27	.69977	2.77	.74643
1.28	.54367	1.78	.63652	2.28	.70085	2.78	.74723
1.29	.54591	1.79	.63804	2.29	.70193	2.79	.74802
1.30	.54812	1.80	.63955	2.30	.70300	2.80	.74881
1.31	.55033	1.81	.64105	2.31	.70406	2.81	.74959
1.32	.55251	1.82	.64254	2.32	.70512	2.82	.75037
1.33	.55468	1.83	.64402	2.33	.70617	2.83	.75114
1.34	.55683	1.84	.64548	2.34	.70721	2.84	.75191
1.35	.55896	1.85	.64694	2.35	.70825	2.85	.75268
1.36	.56108	1.86	.64838	2.36	.70928	2.86	.75344
1.37	.56317	1.87	.64982	2.37	.71030	2.87	.75420
1.38	.56525	1.88	.65125	2.38	.71132	2.88	.75495
1.39	.56732	1.89	.65266	2.39	.71233	2.89	.75570
1.40	.56937	1.90	.65407	2.40	.71333	2.90	.75644
1.41	.57140	1.91	.65546	2.41	.71433	2.91	.75718
1.42	.57342	1.92	.65685	2.42	.71532	2.92	.75791
1.43	.57542	1.93	.65822	2.43	.71631	2.93	.75864
1.44	.57740	1.94	.65959	2.44	.71728	2.94	.75937
1.45	.57937	1.95	.66094	2.45	.71826	2.95	.76009
1.46	.58132	1.96	.66229	2.46	.71922	2.96	.76081
1.47	.58326	1.97	.66363	2.47	.72018	2.97	.76152
1.48	.58518	1.98	.66495	2.48	.72114	2.98	.76224
1.49	.58709	1.99	.66627	2.49	.72209	2.99	.76295
1.50	.58898	2.00	.66758	2.50	.72303	3.00	.76365

a/ℓ	P_c	a/ℓ	P_c	a/ℓ	P_c	a/ℓ	P_c
3.00	.76365	3.50	.79440	4.00	.81834	4.50	.83744
3.01	.76435	3.51	.79493	4.01	.81877	4.51	.83779
3.02	.76504	3.52	.79547	4.02	.81919	4.52	.83813
3.03	.76573	3.53	.79600	4.03	.81961	4.53	.83847
3.04	.76642	3.54	.79653	4.04	.82003	4.54	.83880
3.05	.76711	3.55	.79706	4.05	.82044	4.55	.83914
3.06	.76779	3.56	.79758	4.06	.82086	4.56	.83948
3.07	.76846	3.57	.79810	4.07	.82127	4.57	.83981
3.08	.76914	3.58	.79862	4.08	.82168	4.58	.84014
3.09	.76980	3.59	.79914	4.09	.82209	4.59	.84048
3.10	.77047	3.60	.79966	4.10	.82250	4.60	.84081
3.11	.77113	3.61	.80017	4.11	.82291	4.61	.84113
3.12	.77179	3.62	.80068	4.12	.82331	4.62	.84146
3.13	.77244	3.63	.80118	4.13	.82371	4.63	.84179
3.14	.77309	3.64	.80169	4.14	.82411	4.64	.84211
3.15	.77374	3.65	.80219	4.15	.82451	4.65	.84244
3.16	.77439	3.66	.80269	4.16	.82491	4.66	.84276
3.17	.77503	3.67	.80319	4.17	.82530	4.67	.84308
3.18	.77566	3.68	.80368	4.18	.82570	4.68	.84340
3.19	.77630	3.69	.80417	4.19	.82609	4.69	.84372
3.20	.77693	3.70	.80466	4.20	.82648	4.70	.84403
3.21	.77756	3.71	.80515	4.21	.82687	4.71	.84435
3.22	.77818	3.72	.80564	4.22	.82725	4.72	.84467
3.23	.77880	3.73	.80612	4.23	.82764	4.73	.84498
3.24	.77942	3.74	.80660	4.24	.82802	4.74	.84529
3.25	.78003	3.75	.80708	4.25	.82841	4.75	.84560
3.26	.78064	3.76	.80755	4.26	.82879	4.76	.84591
3.27	.78125	3.77	.80803	4.27	.82916	4.77	.84622
3.28	.78186	3.78	.80850	4.28	.82954	4.78	.84653
3.29	.78246	3.79	.80897	4.29	.82992	4.79	.84683
3.30	.78305	3.80	.80944	4.30	.83029	4.80	.84714
3.31	.78365	3.81	.80990	4.31	.83066	4.81	.84744
3.32	.78424	3.82	.81036	4.32	.83104	4.82	.84774
3.33	.78483	3.83	.81082	4.33	.83140	4.83	.84805
3.34	.78542	3.84	.81128	4.34	.83177	4.84	.84835
3.35	.78600	3.85	.81174	4.35	.83213	4.85	.84865
3.36	.78658	3.86	.81219	4.36	.83250	4.86	.84894
3.37	.78716	3.87	.81265	4.37	.83286	4.87	.84924
3.38	.78773	3.88	.81310	4.38	.83322	4.88	.84954
3.39	.78830	3.89	.81355	4.39	.83358	4.89	.84983
3.40	.78887	3.90	.81399	4.40	.83394	4.90	.85012
3.41	.78944	3.91	.81444	4.41	.83430	4.91	.85042
3.42	.79000	3.92	.81488	4.42	.83465	4.92	.85071
3.43	.79056	3.93	.81532	4.43	.83501	4.93	.85100
3.44	.79111	3.94	.81576	4.44	.83536	4.94	.85129
3.45	.79167	3.95	.81619	4.45	.83571	4.95	.85157
3.46	.79222	3.96	.81663	4.46	.83606	4.96	.85186
3.47	.79277	3.97	.81706	4.47	.83641	4.97	.85215
3.48	.79331	3.98	.81749	4.48	.83675	4.98	.85243
3.49	.79386	3.99	.81792	4.49	.83710	4.99	.85272
3.50	.79440	4.00	.81834	4.50	.83744	5.00	.85300

where

$$\mu = \vec{\Omega} \cdot \vec{n}_1 = \cos\theta \tag{2}$$

Changing from α to R as coordinate we use

$$d\alpha = \left(\frac{\partial \alpha}{\partial R}\right)_z dR \tag{3}$$

The cosine law gives:

$$L^2 = R^2 + a^2 - 2aR\mu \tag{4}$$

$$L^2 = a^2 + z^2 \tag{5}$$

Subtracting (5) from (4) and solving for μ gives

$$\mu = \frac{R^2 - z^2}{2aR} \tag{6}$$

Similarly

$$Y^2 = 2a^2 - 2a^2\nu \tag{7}$$

where $\cos\alpha = \nu$

and

$$Y^2 = R^2 - z^2 \tag{8}$$

Hence:

$$1 - \nu = \frac{R^2 - z^2}{2a^2} \tag{9}$$

$$\left(\frac{\partial \alpha}{\partial R}\right)_z = \frac{R}{a^2\sqrt{1 - \nu^2}} \tag{10}$$

We have:

$$\nu - 1 = \frac{z^2 - R^2}{2a^2} \qquad (11)$$

$$\nu + 1 = \frac{z^2 - R^2}{2a^2} + 2 \qquad (12)$$

Therefore

$$\sqrt{1 - \nu^2} = \sqrt{-(\nu - 1)(\nu + 1)} = \frac{1}{2a^2}\sqrt{R^2 - z^2}\,\sqrt{4a^2 + z^2 - R^2} \qquad (13)$$

and

$$\left(\frac{\partial \alpha}{\partial R}\right)_z = \frac{2R}{\sqrt{R^2 - z^2}\,\sqrt{4a^2 + z^2 - R^2}} \qquad (14)$$

which gives:

$$d\Omega = \frac{R^2 - z^2}{R^2\sqrt{R^2 - z^2}\,\sqrt{4a^2 + z^2 - R^2}}\, dRdz \qquad (15)$$

Hence:

$$\phi(R) = \frac{\int_{R_s = R} \mu \frac{d}{dR}\, dS}{\int \vec{\Omega}\cdot\vec{n}_1\, d\Omega\, dS} = \frac{S}{\pi S}\int \mu \frac{d\Omega}{dR} \qquad (16)$$

$$= \frac{1}{\pi}\int \frac{(R^2 - z^2)^{3/2}}{(2aR)R^2\sqrt{4a^2 + z^2 - R^2}}\, dz$$

or

$$\phi(R) = \frac{2\cdot 2}{2a\pi R^3}\int_{z_o}^{R} \frac{(R^2 - z^2)^{3/2}}{\sqrt{4a^2 + z^2 - R^2}}\, dz \qquad (17)$$

where

$$z_o = \begin{cases} \sqrt{R^2 - 4a^2} & R > 2a \\ 0 & R < 2a \end{cases} \qquad (18)$$

In (17) the two factors of 2 come from the facts that the solid angle subtended may be either above or below the point on the cylinder that is being considered and that α runs from 0 to 2π and not from 0 to π.

Introducing a new variable x into (17) where

$$x = \frac{\sqrt{R^2 - z^2}}{2a} \tag{19}$$

gives:

$$\phi(R) = \frac{16a^2}{\pi R^3} \int_0^{x_o} \frac{x^4 \, dx}{\sqrt{\frac{R^2}{4a^2} - x^2} \sqrt{1 - x^2}} \tag{20}$$

where

$$x_o = \begin{cases} 1 & R > 2a \\ R/2a & R < 2a \end{cases} \tag{21}$$

With (20) we find on carrying out the integrals

$$\left.\begin{aligned} &\int_0^\infty \phi(R)dR = 1 \\ &\int_0^\infty R \, \phi(R)dR = 2a = \frac{4V}{S} \end{aligned}\right\} \quad \text{in agreement with earlier considerations} \tag{22}$$

Moreover:

$$(R^2)_{av} = \int_0^\infty R^2 \, \phi(R)dR = 16 \, a^2/3 \tag{23}$$

With $R_{av} = 2a$, from (22), (10.1-10) gives

$$P_o = \frac{\ell}{2a} \left\{ 1 - \int_0^\infty e^{-R/\ell} \phi(R) \, dR \right\} \tag{24}$$

Using (20) we have

$$\int_0^\infty e^{-R/\ell}\phi(R)dR = \frac{16a^2}{\pi}\int_0^{2a}\frac{e^{-R/\ell}dR}{R^3}\int_0^{R/2a}\frac{x^4dx}{\sqrt{1-x^2}\sqrt{\frac{R^2}{4a^2}-x^2}}$$

$$+ \frac{16a^2}{\pi}\int_{2a}^\infty\frac{e^{-R/\ell}dR}{R^3}\int_0^1\frac{x^4dx}{\sqrt{1-x^2}\sqrt{\frac{R^2}{4a^2}-x^2}} \qquad (25)$$

Introducing $y = R/2a$ as a variable in (25) and changing the order of integration gives:

$$\int_0^\infty e^{-R/\ell}\phi(R)dR = \frac{4}{\pi}\int_0^1\frac{x^4dx}{\sqrt{1-x^2}}\int_1^\infty\frac{e^{-(2a/\ell)y}}{y^3\sqrt{y^2-x^2}}dy$$

$$+ \frac{4}{\pi}\int_0^1\frac{x^4dx}{\sqrt{1-x^2}}\int_x^1\frac{e^{-(2a/\ell)y}}{y^3\sqrt{y^2-x^2}}dy \qquad (26)$$

$$= \frac{4}{\pi}\int_0^1\frac{x^4dx}{\sqrt{1-x^2}}\int_x^\infty\frac{e^{-(2a/\ell)y}}{y^3\sqrt{y^2-x^2}}dy$$

The integration in (26) has been performed by Inglis* giving for P_o:

$$P_o = \frac{z}{3}\frac{a}{\ell}\left\{2\left[\frac{a}{\ell}K_1\left(\frac{a}{\ell}\right)I_1\left(\frac{a}{\ell}\right) + K_o\left(\frac{a}{\ell}\right)I_o\left(\frac{a}{\ell}\right) - 1\right.\right.$$

$$\left.+ \frac{K_1\left(\frac{a}{\ell}\right)I_1\left(\frac{a}{\ell}\right)}{a/\ell} - K_o\left(\frac{a}{\ell}\right)I_1\left(\frac{a}{\ell}\right) + K_1\left(\frac{a}{\ell}\right)I_o\left(\frac{a}{\ell}\right)\right\} \qquad (27)$$

where K_n, I_n are the Bessel Functions as defined and tabulated by Watson.**

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*D. Inglis - Los Alamos Classified Report LA-26, page 8.

**G. N. Watson - "Theory of Bessel Functions", Cambridge University Press, 1945.

For radii small compared to a mean free path (27) may be expanded as:

$$P_o = 1 - \frac{4}{3}\frac{a}{\ell} + \frac{1}{2}\left(\frac{a}{\ell}\right)^2 \log\left(\frac{2\ell}{a}\right) + \frac{1}{2}\left(\frac{a}{\ell}\right)^2\left(\frac{5}{4} - \gamma\right) \quad (28)$$

where γ = Euler's constant = 0.577215665 ...

(28) gives P_o correct to five significant figures provided $a/\ell < 0.1$.

For "a" large compared to ℓ the asymptotic expansion of the Bessel functions give for P_o:

$$P_o = \frac{1}{2(\frac{a}{\ell})} - \frac{3}{32(\frac{a}{\ell})^3} \quad (29)$$

in agreement with (10.1-14). (29) is correct to five significant figures provided $a/\ell > 6$.

Table 4 gives the collision probability P_c as a function of a/ℓ. P_c is plotted in Figure 11.

10.4 The Hemisphere

Chord distributions for the hemisphere, oblate spheroid, and oblate hemispheroid have been obtained by Dirac, Fuchs, Peierls, and Preston*. We omit the rather complicated analysis and merely quote results.

Let $x = R/R_m$ (1)

and

$$\psi(x)dx = \phi(R)dR \quad (2)$$

where R_m is the largest chord that can be drawn in the body. Equation (10.1-7) becomes:

$$P_o = \frac{\ell}{R_m}\frac{1}{x_{av}}\int_0^1 \left[1 - e^{-\frac{x}{\ell/R_m}}\right]\psi(x)dx \quad (3)$$

*M.S.D.5 - "Applications to the Oblate Spheroid, Hemi-sphere and Oblate Hemispheroid" by Dirac et al.

TABLE 4

VALUES OF COLLISION PROBABILITY P_c

$(P_c = 1 - P_o)$

FOR INFINITE CYLINDER OF RADIUS "a"

Note: This table gives values for $0.1 \leq a \leq 6.0$ calculated from

$$P_c = 1 - \frac{2a}{3\ell}\left\{2\left[\left(\frac{a}{\ell}\right)(K_1I_1 + K_oI_o) - 1\right] + \frac{K_1I_1}{(a/\ell)} - K_oI_1 + K_1I_o\right\}$$

where the K_n and I_n are the functions defined in Watson's "Bessel functions" (page 698) and the arguments of K_n and I_n are "a/ℓ"

for $(a/\ell) \geq 6$ the following expansion is good to five significant figures

$$P_c = 1 - \frac{\ell}{2a} + \frac{3}{32}\left(\frac{\ell}{a}\right)^3$$

for $(a/\ell) \leq 0.08$ the following expansion is good to five significant figures

$$P_c = \frac{4a}{3\ell} + \frac{1}{2}\left(\frac{a}{\ell}\right)^2\left(\gamma - \frac{5}{4}\right) + \frac{1}{2}\left(\frac{a}{\ell}\right)^2 \log\frac{a}{2\ell}$$

where $\gamma = 0.577215665$

TABLE 4

a/l	P_c	a/l	P_c	a/l	P_c	a/l	P_c	a/l	P_c	a/l	P_c
0	.00000	1.02	.59826	2.02	.76562	3.02	.83816	4.02	.87714	5.02	.90116
.02	.02561	1.04	.60354	2.04	.76765	3.04	.83917	4.04	.87773	5.04	.90155
.04	.04967	1.06	.60870	2.06	.76966	3.06	.84017	4.06	.87832	5.06	.90193
.06	.07248	1.08	.61375	2.08	.77163	3.08	.84116	4.08	.87890	5.08	.90231
.08	.09421	1.10	.61869	2.10	.77357	3.10	.84214	4.10	.87947	5.10	.90268
.10	.11498	1.12	.62352	2.12	.77548	3.12	.84310	4.12	.88004	5.12	.90306
.12	.13487	1.14	.62825	2.14	.77736	3.14	.84405	4.14	.88061	5.14	.90343
.14	.15396	1.16	.63287	2.16	.77921	3.16	.84500	4.16	.88117	5.16	.90381
.16	.17231	1.18	.63740	2.18	.78103	3.18	.84593	4.18	.88173	5.18	.90417
.18	.18996	1.20	.64183	2.20	.78282	3.20	.84685	4.20	.88228	5.20	.90453
.20	.20697	1.22	.64616	2.22	.78459	3.22	.84776	4.22	.88282	5.22	.90489
.22	.22336	1.24	.65041	2.24	.78633	3.24	.84866	4.24	.88336	5.24	.90525
.24	.23918	1.26	.65457	2.26	.78805	3.26	.84955	4.26	.88389	5.26	.90560
.26	.25446	1.28	.65864	2.28	.78974	3.28	.85043	4.28	.88443	5.28	.90595
.28	.26923	1.30	.66263	2.30	.79140	3.30	.85130	4.30	.88495	5.30	.90631
.30	.28351	1.32	.66654	2.32	.79304	3.32	.85216	4.32	.88547	5.32	.90665
.32	.29733	1.34	.67038	2.34	.79465	3.34	.85301	4.34	.88598	5.34	.90700
.34	.31070	1.36	.67413	2.36	.79624	3.36	.85385	4.36	.88650	5.36	.90734
.36	.32366	1.38	.67781	2.38	.79781	3.38	.85468	4.38	.88701	5.38	.90768
.38	.33621	1.40	.68142	2.40	.79936	3.40	.85550	4.40	.88751	5.40	.90802
.40	.34838	1.42	.68495	2.42	.80088	3.42	.85632	4.42	.88801	5.42	.90835
.42	.36019	1.44	.68842	2.44	.80239	3.44	.85712	4.44	.88850	5.44	.90868
.44	.37164	1.46	.69182	2.46	.80387	3.46	.85792	4.46	.88899	5.46	.90902
.46	.38276	1.48	.69516	2.48	.80532	3.48	.85870	4.48	.88948	5.48	.90934
.48	.39356	1.50	.69843	2.50	.80677	3.50	.85948	4.50	.88996	5.50	.90967
.50	.40405	1.52	.70164	2.52	.80818	3.52	.86025	4.52	.89044	5.52	.90999
.52	.41424	1.54	.70479	2.54	.80958	3.54	.86101	4.54	.89091	5.54	.91031
.54	.42414	1.56	.70788	2.56	.81096	3.56	.86177	4.56	.89138	5.56	.91063
.56	.43377	1.58	.71091	2.58	.81233	3.58	.86251	4.58	.89184	5.58	.91095
.58	.44314	1.60	.71389	2.60	.81367	3.60	.86325	4.60	.89230	5.60	.91126
.60	.45225	1.62	.71681	2.62	.81500	3.62	.86398	4.62	.89276	5.62	.91157
.62	.46112	1.64	.71968	2.64	.81630	3.64	.86471	4.64	.89321	5.64	.91188
.64	.46975	1.66	.72250	2.66	.81759	3.66	.86542	4.66	.89366	5.66	.91219
.66	.47816	1.68	.72526	2.68	.81886	3.68	.86613	4.68	.89411	5.68	.91250
.68	.48634	1.70	.72798	2.70	.82012	3.70	.86683	4.70	.89455	5.70	.91280
.70	.49432	1.72	.73065	2.72	.82136	3.72	.86752	4.72	.89499	5.72	.91310
.72	.50209	1.74	.73327	2.74	.82258	3.74	.86821	4.74	.89542	5.74	.91340
.74	.50966	1.76	.73585	2.76	.82379	3.76	.86889	4.76	.89585	5.76	.91369
.76	.51704	1.78	.73838	2.78	.82498	3.78	.86956	4.78	.89628	5.78	.91399
.78	.52424	1.80	.74087	2.80	.82616	3.80	.87023	4.80	.89671	5.80	.91429
.80	.53126	1.82	.74331	2.82	.82732	3.82	.87089	4.82	.89713	5.82	.91457
.82	.53811	1.84	.74572	2.84	.82847	3.84	.87154	4.84	.89755	5.84	.91487
.84	.54479	1.86	.74808	2.86	.82960	3.86	.87219	4.86	.89796	5.86	.91515
.86	.55131	1.88	.75040	2.88	.83071	3.88	.87283	4.88	.89837	5.88	.91544
.88	.55767	1.90	.75269	2.90	.83182	3.90	.87346	4.90	.89878	5.90	.91572
.90	.56389	1.92	.75493	2.92	.83291	3.92	.87409	4.92	.89919	5.92	.91600
.92	.56996	1.94	.75714	2.94	.83399	3.94	.87471	4.94	.89959	5.94	.91628
.94	.57588	1.96	.75931	2.96	.83505	3.96	.87533	4.96	.89999	5.96	.91656
.96	.58167	1.98	.76145	2.98	.83610	3.98	.87594	4.98	.90038	5.98	.91684
.98	.58733	2.00	.76355	3.00	.83714	4.00	.87654	5.00	.90077	6.00	.91711
1.00	.59285										

For a hemisphere of radius a:

$$R_m = 2a$$

and

$$\psi(z) = \frac{2}{3\pi} \left\{ \sqrt{1 - x^2}\ (3 + \frac{1}{2x^2}) + \frac{1}{2x^3} \cos^{-1} x \right\} \qquad (x > \tfrac{1}{2}) \tag{4}$$

$$= \frac{2}{3\pi} \left\{ \sqrt{1 - x^2}\ (3 + \frac{1}{2x^2}) + \frac{1}{2x^3} \cos^{-1} x - \frac{\pi}{4} (\frac{1}{x^3} - 16x) \right\} \qquad (x < \tfrac{1}{2}) \tag{5}$$

The expansion for P_o in the region $a/\ell << 1$ obtained by expanding the exponential in (3) is

$$P_o = 1 - \frac{9}{16} (\frac{a}{\ell}) + (\frac{a}{\ell})^2 \frac{(32 - 3\pi)}{30\pi} - \frac{1}{12} (\frac{a}{\ell})^3 + \frac{1}{40\pi} (\frac{a}{\ell})^4 \frac{(1024-15\pi)}{315} \cdots \tag{6}$$

For $a/\ell >> 1$ we have the expansion (10.1-12). Since

$$R_{av} = \frac{8}{9} a; \quad \phi(0) = \frac{1}{a} \frac{8}{9\pi}; \quad \phi'(0) = \frac{4}{3a^2} \tag{7}$$

$$P_o \approx \frac{9}{8} \frac{\ell}{a} \left\{ 1 - \frac{8}{9\pi} \frac{\ell}{a} - \frac{2}{3} (\frac{\ell}{a})^2 \right\} \qquad a/\ell >> 1 \tag{8}$$

The linear correction term is, of course, to be expected since the hemisphere has an edge and so does not have $\phi(0) = 0$.

A somewhat more accurate expansion in the region $a/\ell >> 1$ is obtained by slightly modifying the procedure which led to (10.1-12). Instead of expanding $\phi(R)$ around R=0 and integrating to 2a one can integrate this to a and then to integrate the remainder of the way use the expansion of $\phi(R)$ around R=a. This gives

$$P_o = 1.1250\left(\frac{\ell}{a}\right) - \left(\frac{\ell}{a}\right)^2\left(1 - e^{-a/\ell}\right)\left(0.31831 + 1.01586\, e^{-a/\ell}\right)$$

$$- \left(\frac{\ell}{a}\right)^3 \left\{1 - e^{-a/\ell}\left[1 + \frac{a}{\ell}\right]\right\}\left\{0.75000 - 2.3614\, e^{-a/\ell}\right\}$$

$$- \left(\frac{\ell}{a}\right)^4 \left\{1 - e^{-a/\ell}\left[1 + \frac{a}{\ell} + \frac{1}{2}\left(\frac{a}{\ell}\right)^2\right]\right\}\left\{- 0.080574 + 8.8492\, e^{-a/\ell}\right\} \qquad (9)$$

$$- \left(\frac{\ell}{a}\right)^5 \left\{1 - e^{-a/\ell}\left[1 + \frac{a}{\ell} + \frac{1}{2}\left(\frac{a}{\ell}\right)^2 + \frac{1}{6}\left(\frac{a}{\ell}\right)^3\right]\right\}\left\{0 - 45.162\, e^{-a/\ell}\right\}$$

Equations (6) and (9) give P_o to reasonable accuracy for all a/ℓ except in the region $1 < a/\ell < 3$. Table (5) gives escape probability as a function of a/ℓ determined by numerical integration in the intermediate region. Fig. 11 contains a plot of P_c vs. a/ℓ for the hemisphere. Fig. 14 and 15 contain plots of P_o/P_{os} vs. P_{os} and P_c/P_{cs} vs. P_{os} where P_{cs} and P_{os} are collision and escape probability for a sphere of the same volume as the hemisphere.

10.5 The Oblate Spheroid

Consider an oblate spheroid with major and minor axes and and b:

If

$$\epsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2} \qquad (1)$$

and

$$F(\epsilon) = 1 + (1 - \epsilon^2)\,\frac{\tanh^{-1}\epsilon}{\epsilon} \qquad (2)$$

it can be shown* that

$$\Psi(x) = x\left\{\frac{1}{(1 - \epsilon^2)F(\epsilon)} + \frac{3}{2}\right\} \qquad \left(x < \sqrt{1 - \epsilon^2}\right) \qquad (3)$$

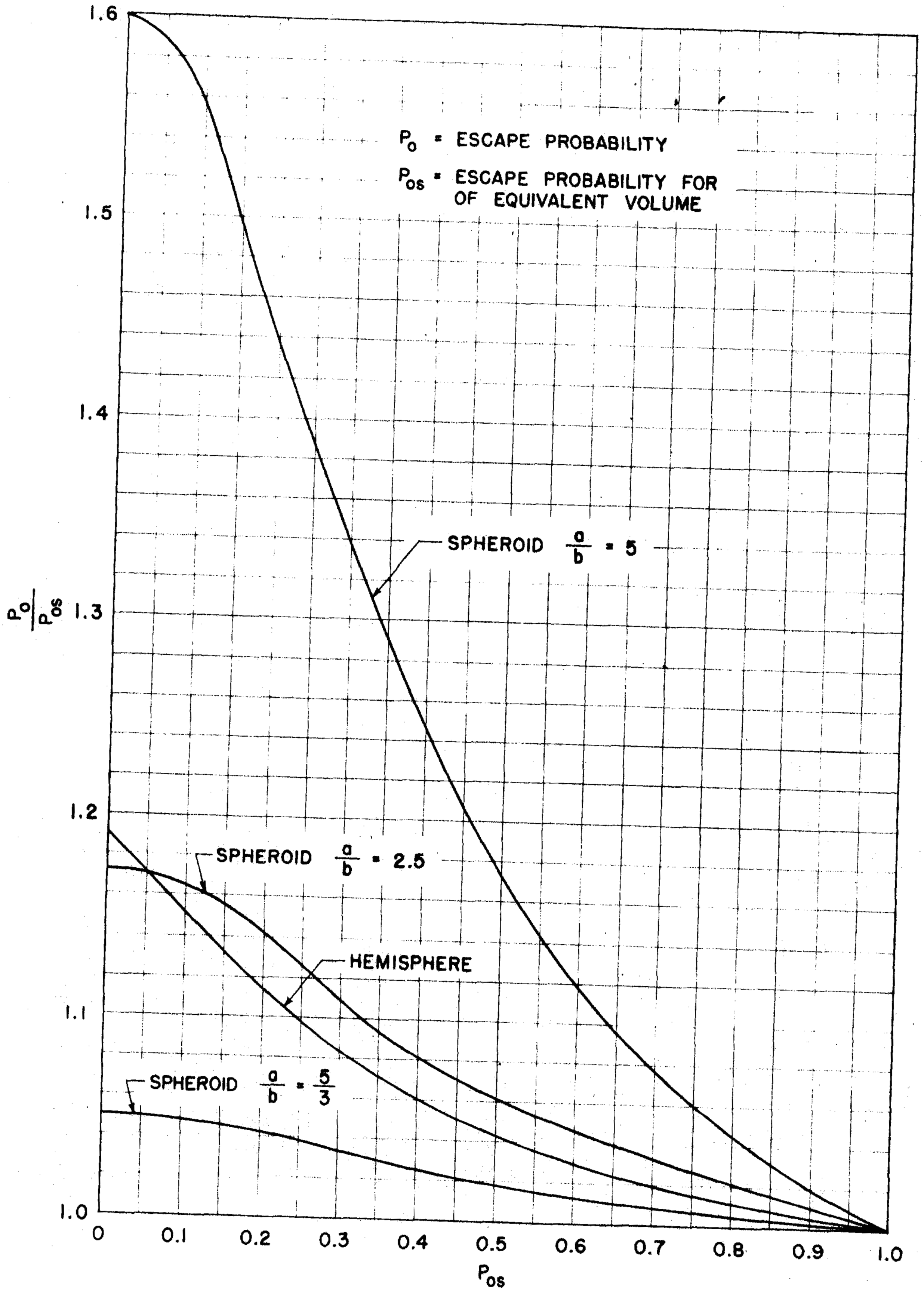
$$\Psi(x) = \frac{1 - \epsilon^2}{\epsilon F(\epsilon)}\left\{\frac{\sqrt{1 - x^2}}{x^3}\left(1 + \frac{3}{2}x^2\right) + \frac{3}{2}\, x \tanh^{-1}\left[\sqrt{1 - x^2}\right]\right\} \quad \left(x > \sqrt{1 - \epsilon^2}\right) \qquad (4)$$

*M.S.D.5 - "Applications to the Oblate Spheroid, Hemi-sphere and Oblate Hemispheroid" by Dirac et al.

TABLE 5

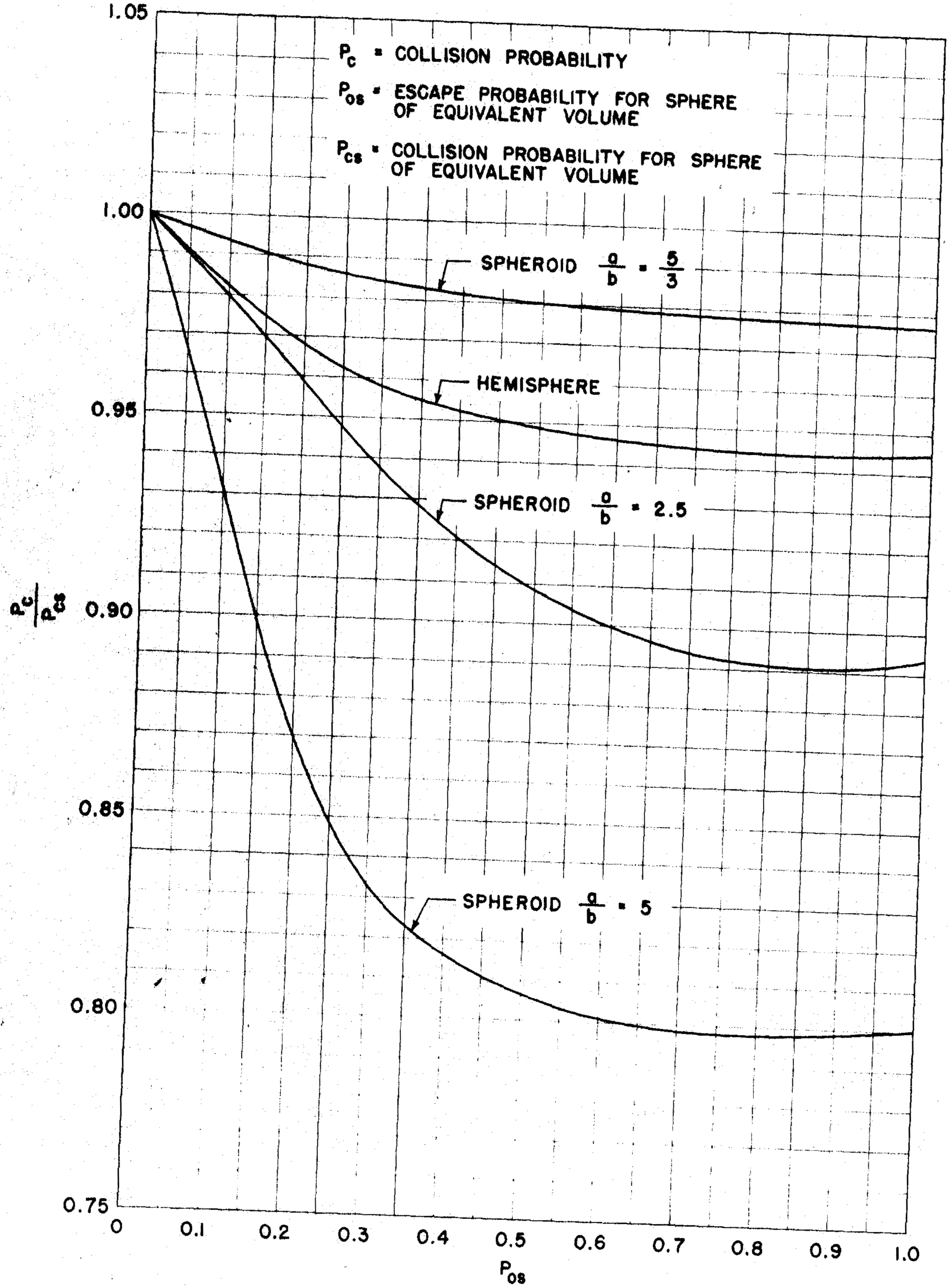
ESCAPE PROBABILITY FOR HEMISPHERE

a/ℓ	P_o
0	1.000
.2	0.896
.5	0.770
1.0	0.613
1.5	0.503
2.0	0.423
3.0	0.317
4.0	0.251
5.0	0.207
6.0	0.175
7.0	0.152
8.0	0.134
9.0	0.120
10.0	0.109



1.6
1.5
1.4
1.3
1.2
1.1
1.0
P_o = ESCAPE PROBABILITY
P_{os} = ESCAPE PROBABILITY FOR OF EQUIVALENT VOLUME
SPHEROID $\frac{a}{b}$ = 5
SPHEROID $\frac{a}{b}$ = 2.5
HEMISPHERE
SPHEROID $\frac{a}{b}$ = $\frac{5}{3}$
P_o / P_{os}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
P_{os}

Fig. 14



P_c = COLLISION PROBABILITY
P_{os} = ESCAPE PROBABILITY FOR SPHERE OF EQUIVALENT VOLUME
P_{cs} = COLLISION PROBABILITY FOR SPHERE OF EQUIVALENT VOLUME
SPHEROID $\frac{a}{b} = \frac{5}{3}$
HEMISPHERE
SPHEROID $\frac{a}{b} = 2.5$
SPHEROID $\frac{a}{b} = 5$
P_c/P_{cs}
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
P_{os}

Fig. 15

Hence:

$$(x^2)_{av} = \int_0^1 x^2 \; \psi(x)dx = \frac{1-\epsilon^2}{F(\epsilon)} \quad \frac{\tanh^{-1}\epsilon}{\epsilon} \tag{5}$$

$$(R^2)_{av} = \frac{4a^2(1-\epsilon^2)}{F(\epsilon)} \quad \frac{\tanh^{-1}\epsilon}{\epsilon} \tag{6}$$

Since

$$R_{av} = \frac{4V}{S} = \frac{8}{3}\frac{b}{F(\epsilon)} \tag{7}$$

(10.1-9) gives for $a/\ell << 1$

$$P_o \approx 1 - \frac{3}{4}\frac{b}{\ell} \quad \frac{\tanh^{-1}\epsilon}{\epsilon} \text{ or } P_c \approx \frac{3}{4}\frac{b}{\ell} \quad \frac{\tanh^{-1}\epsilon}{\epsilon} \tag{8}$$

For $b/\ell >> 1$ we have as a first approximation using (10.1-13) and (7)

$$P_o = \frac{3}{8} F(\epsilon) \frac{\ell}{b} \tag{9}$$

Of particular interest is the dependence of escape probability on shape. A measure of this dependence is the ratio of the escape probability (or collision probability) for a body to that of a sphere of the same volume.

From (8) and (9) we obtain

$$P_c/P_{cs} = (1-\epsilon^2)^{1/3} \frac{\tanh^{-1}\epsilon}{\epsilon} \qquad a/\ell << 1 \tag{10}$$

and

$$P_o/P_{os} = \frac{1}{2} \frac{F(\epsilon)}{(1-\epsilon^2)^{1/3}} \qquad b/\ell >> 1 \tag{11}$$

If $\epsilon << 1$ (10) and (11) give on expansion

$$P_c/P_{cs} = 1 - \frac{\epsilon^4}{45} \left\{ 1 + \frac{64}{63}\epsilon^2 + \ldots \right\} \tag{12}$$

and

$$P_o/P_{os} = 1 + \frac{2}{45}\,\epsilon^4 \left\{1 + \frac{68}{63}\,\epsilon^2 + \dots\right\} \qquad (13)$$

It should be noted that (12) and (13) show that the difference in escape probabilities between an oblate spheroid and a sphere of the same volume is proportional to ϵ^4. Thus it is only for eccentricities extremely close to one that the spheroid will have an escape probability differing appreciably from that of the equivalent sphere.

In the other extreme of $1 - \epsilon \ll 1$ (10) and (11) give

$$P_c/P_{cs} = \left(\frac{b}{a}\right)^{2/3} \log 2a/b \qquad (\ell \gg a \gg b) \qquad (14)$$

$$P_o/P_{os} = \frac{1}{2}\left(\frac{a}{b}\right)^{2/3} \qquad (a \gg b \gg \ell) \qquad (15)$$

Fig. 14 and 15 show P_o/P_{os} vs. P_{os} and P_c/P_{cs} vs. P_{os} for $a/b = 5$, 2.5, and 1.67. It should be noted that in Fig. 15 the scale has been expanded to magnify the small deviations of the collision probabilities from that of the equivalent sphere.

10.6 Discussion of Results

To facilitate the calculation of escape probabilities for bodies whose dimensions are small compared to the mean free path the quantity

$$Q = \frac{1}{2}\,\frac{(R^2)_{av}}{(R_{av})^2} \qquad (1)$$

for various special shapes are collected in Table 6.

Table 6

Shape	$Q = (R^2)_{av}/2(R_{av})^2$
Sphere	0.5625
Hemisphere	0.6328
Oblate Spheroid	$0.28125\ F(\epsilon)\ \frac{\tanh^{-1}\epsilon}{\epsilon}$
$\epsilon << 1$	$0.5625\ \left\{1 + \epsilon^4/45\right\}$
$b/a << 1$	$0.28125 \log\left(\frac{2a}{b}\right)$
Tetrahedron	0.7915
Infinite Cylinder with Circular Cross-section	0.6667
Infinite Cylinder with Square Cross-section	0.7435
Infinite Cylinder with Equilateral Triangle Cross-section	0.824
Infinite Cylinder with Regular Hexagon Cross-section	0.6985

(10.1-9) tells us that for small bodies

$$P_o \approx 1 - \frac{Q\,R_{av}}{\ell} = 1 - \frac{4V}{S}Q \qquad (2)$$

Those values of Q not contained in the work of the preceding sections were obtained from a paper by Behrens*.

*D. J. Behrens: Proc. Phys. Soc. A, V. LXII; P. 607, (1949) In section (10.5) it was seen that for a fixed volume the escape probabilities are a very insensitive function of the shape. The important shape dependence must then be obtained by considering bodies of the same volume. From (2), though, we see that Q compares collision probabilities for bodies of the same R_{av}. The more relevant shape factor is Q' where

$$Q' = \frac{QR_{av}}{V^{1/3}} \tag{3}$$

in terms of which we have for small bodies

$$P_c = Q' \frac{V^{1/3}}{\ell} \tag{4}$$

Table 7 gives Q'/Q'_{sphere} for the hemisphere, tetrahedron and oblate spheroid.

TABLE 7

Shape	Q'/Q'_{sphere} $Q' = QR_{av}/V^{1/3}$	α/α_{sphere} $\alpha = S/4V^{2/3}$
Hemisphere	0.9449	1.1905
Oblate Spheroid	$(1-\epsilon^2)^{1/3}(\tanh^{-1}\epsilon)/\epsilon$	$1/2\ (1-\epsilon^2)^{1/3}\ F(\epsilon)$
$\epsilon \ll 1$	$1-\epsilon^4/45$	$1+\frac{2}{45}\epsilon^4$
$b/a \ll 1$	$(b/a)^{2/3}\log(2a/b)$	$1/2\ (a/b)^{2/3}$
Tetrahedron	0.9444	1.4899

To compare the escape probabilities of bodies of dimensions large compared to ℓ it is useful to introduce α defined by:

$$\alpha = V^{1/3}/R_{av} = \frac{S}{4V^{2/3}} \qquad (5)$$

In terms of α (10.1-12) gives for large bodies

$$P_o \approx \alpha^{\ell/V^{1/3}} \qquad (6)$$

In Table 7 the quantity α/α_{sphere} is given for hemisphere, oblate spheroid, and tetrahedron.

From Table 7 we see that, for a given volume, escape probabilities are rather shape insensitive. Thus we see that for small bodies the collision probability for hemisphere and tetrahedron is but 5% less than that for a sphere of the same volume. For spheroids of small eccentricity the deviation from the sphere collision probability is proportional to ϵ^4. For large bodies Table 8 shows the shape to be somewhat more important. Fortunately, the escape probability is just the inverse of R_{av}/ℓ. This last is, however, rather simple to evaluate for even complicated bodies.

C. ONE VELOCITY THEORY OF NEUTRON DIFFUSION

III. Equations for a General Medium

11. The Transport Equation and Some of Its Properties

Previously we considered only the case of pure absorption. On collision the neutrons disappeared. A medium in which such propagation takes place is entirely characterized by the cross-section $\sigma(\vec{r})$. We now generalize by considering a medium in which a collision can be followed by fission, scattering, or absorption. In addition to $\sigma(\vec{r})$ such a medium is described by the parameters $c(\vec{r})$ and $f(\vec{\Omega}, \vec{\Omega}', \vec{r})$.

$c(\vec{r})$ is the average number of secondaries emitted after a collision occurs at $\vec{r}$. In the one velocity theory which we are considering, the proportions of fission, scattering, and absorption which give rise to this average value are immaterial. Only the net effect is significant. For pure absorption (the previously considered case) $c = 0$. $c = 1$ results from pure scattering. A medium which can fission and absorb can also have a "c" of unity. $c > 1$ corresponds to a medium in which each collision is followed, on the average, by the emission of more than one neutron. Such a medium is predominantly multiplying.

$f(\vec{\Omega}, \vec{\Omega}', \vec{r})d\Omega'$ is the fraction of neutrons emitted in direction $\vec{\Omega}$ following the collision at $\vec{r}$ of a neutron travelling within $d\Omega'$ of the direction $\vec{\Omega}'$. It is normalized so that

$$\int f(\vec{\Omega}, \vec{\Omega}')\, d\Omega' = \int f(\vec{\Omega}, \vec{\Omega}')\, d\Omega = 1 \tag{1}$$

Since $v\vec{\sigma}(r)\, \psi(\vec{r}, \vec{\Omega}')\, d\Omega'$ is the number of collisions per unit volume and unit time at $\vec{r}$ of neutrons travelling within $d\Omega'$ of $\vec{\Omega}'$

$$c(\vec{r})\, v\, \vec{\sigma}(\vec{r}) \int \psi(\vec{r}, \vec{\Omega}')\, f(\vec{\Omega}, \vec{\Omega}')\, d\Omega' \tag{2}$$

is the number of neutrons of direction $\vec{\Omega}$ being produced at $\vec{r}$ due to such collisions. With the presence of this angular source density the equation of continuity must be modified to read

$$\begin{aligned} v \operatorname{div} \vec{\Omega}\ \psi(\vec{r},\vec{\Omega}) + v\ \sigma(\vec{r})\ \psi(\vec{r},\vec{\Omega}) \\ = q(\vec{r},\vec{\Omega}) + c(\vec{r})\ v\ \sigma(\vec{r}) \int \psi(\vec{r},\vec{\Omega})\ f(\vec{\Omega},\vec{\Omega}')\ d\Omega' \end{aligned} \tag{3}$$

This integro-differential equation is usually referred to as the transport equation. It is linear because we assume that the neutrons collide only with the particles of the relatively dense material medium and not with each other. The transport equation (3) thus corresponds to the linear case of the general kinetic theory Boltzmann equation.

Equation (3) is the same as (5-3) with q replaced by the right hand side of (3). Hence the solution $\psi(\vec{r},\vec{\Omega})$ of (3) satisfied (5-11) (the solution of (5-3)) with this replacement. This is then an integral equation for ψ. In this form, it is readily seen that the results of section (5) concerning boundary conditions may be taken over directly.

The Green's Function of equation (3) subject to the boundary condition of zero incoming angular density satisfies the reciprocity theorem provided f is of the form $f(\vec{\Omega}-\vec{\Omega}')$. (I.e., f depends only on the relative direction between incident and emitted neutron.)

In practice this condition is always satisfied except in some problems of neutron diffusion in single crystals. For such cases the present treatment is not applicable anyway since, for example, σ will depend on the angle of incidence.

To prove the reciprocity theorem let, as usual, $\psi(\vec{r},\vec{\Omega};\ \vec{r}_0,\vec{\Omega}_0)$ be the solution of (3) with q a directional point source at $\vec{r}_0$ emitting in direction $\vec{\Omega}_0$.

Then (3) is:

$$v\vec{\Omega}\cdot\nabla\,\psi(\vec{r},\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o) + v\sigma(\vec{r})\,\psi(\vec{r},\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o)$$

$$= \delta(\vec{r}-\vec{r}_o)\,\delta_2\,(\vec{\Omega}\cdot\vec{\Omega}_o) + vc(\vec{r})\sigma(\vec{r})\int\psi(\vec{r},\vec{\Omega}';\,\vec{r}_o,\vec{\Omega}_o)\,f\,(\vec{\Omega}\cdot\vec{\Omega}')\,d\Omega' \tag{4}$$

and similarly:

$$-v\vec{\Omega}\cdot\nabla\,\psi(\vec{r},-\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1) + v\sigma(\vec{r})\,\psi(\vec{r},-\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1)$$

$$= \delta(\vec{r}-\vec{r}_1)\,\delta_2\,(\vec{\Omega}\cdot\vec{\Omega}_1) + vc(\vec{r})\sigma(\vec{r})\int\psi(\vec{r},-\vec{\Omega}';\,r_1,-\vec{\Omega}_1)\,f\,(\vec{\Omega}\cdot\vec{\Omega}')\,d\Omega' \tag{5}$$

The boundary conditions associated with (4) and (5) are that there are to be no neutrons incident from the outside.

$$\text{i.e.,}\quad \psi(\vec{r}_s,\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o) = \psi(\vec{r}_s,\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1) = 0 \qquad \text{for } (\vec{n}\cdot\vec{\Omega}) < 0 \tag{6}$$

when $\vec{r}_s$ is on the bounding surface of the region in which (4) and (5) are to hold. $\vec{n}$ is the outer normal to this surface at $\vec{r}$. Multiply (4) by $\psi(\vec{r},-\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1)$, (5) by $\psi(\vec{r},\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o)$ and subtract.

Then:

$$v\vec{\Omega}\cdot\nabla\left\{\psi(\vec{r},\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o)\,\psi(\vec{r},-\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1)\right\}$$

$$= \psi(\vec{r},-\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1)\,\delta(\vec{r}-\vec{r}_o)\,\delta_2(\vec{\Omega}\cdot\vec{\Omega}_o) - \psi(\vec{r},\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o)\,\delta(\vec{r}-\vec{r}_1)\,\delta_2(\vec{\Omega}\cdot\vec{\Omega}_o)$$

$$+ vc(\vec{r})\,\sigma(\vec{r})\int\Big\{\psi(\vec{r},-\vec{\Omega};\,\vec{r}_1,-\vec{\Omega}_1)\,f\,(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},\vec{\Omega}';\,\vec{r}_o,\vec{\Omega}_o)$$

$$- \psi(\vec{r},\vec{\Omega};\,\vec{r}_o,\vec{\Omega}_o)\,f\,(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},-\vec{\Omega}';\,\vec{r}_1,-\vec{\Omega}_1)\Big\}\,d\Omega' \tag{7}$$

Integrating (7) over the volume of the body we are considering gives for the left hand side:

$$\int_V v\vec{\Omega}\cdot\nabla\left\{\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\,\psi(\vec{r},-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)\right\}dV$$

$$= \int_S v\vec{n}\cdot\vec{\Omega}\,\psi(\vec{r}_s,\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\,\psi(\vec{r}_s,-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)\,dS \tag{8}$$

where S denotes the bounding surface and V the volume. Since $\vec{n}\cdot\vec{\Omega} < 0$ or $-\vec{n}\cdot\vec{\Omega} < 0$, either $\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)$ or $\psi(\vec{r},-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)$ are zero because of the boundary conditions (6). Hence (8) is zero for all $\vec{\Omega}$. The volume integral of (7) thus becomes:

$$\psi(\vec{r}_0,-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)\;\delta_2(\vec{\Omega}\cdot\vec{\Omega}_0) - \psi(\vec{r}_1,\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\,\delta_2(\vec{\Omega}\cdot\vec{\Omega}_1)$$

$$+\int vc(\vec{r})\,\sigma(\vec{r})\int\left\{\psi(\vec{r},-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)\,f(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},\vec{\Omega}';\vec{r}_0,\vec{\Omega}_0)\right. \tag{9}$$

$$\left.-\,\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\,f(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},-\vec{\Omega}';\vec{r}_1,-\vec{\Omega}_1)\right\}d\Omega'\,dV = 0$$

Integrating (9) qver all Ω gives:

$$\psi(\vec{r}_0,-\vec{\Omega}_0;\vec{r}_1,-\vec{\Omega}_1) - \psi(\vec{r}_1,\vec{\Omega}_1;\vec{r}_0,\vec{\Omega}_0)$$

$$+\int vc(\vec{r})\,\sigma(\vec{r})\iint\left\{\psi(\vec{r},\vec{\Omega}';\vec{r}_0,\vec{\Omega}_0)\,f(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)\right. \tag{10}$$

$$\left.-\,\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\,f(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},-\vec{\Omega}';\vec{r}_1,-\vec{\Omega}_1)\right\}d\Omega\,d\Omega'\,dV = 0$$

Interchanging $\vec{\Omega}$ and $\vec{\Omega}'$ and noting that f is symmetric in the two we see that:

$$\iint\psi(\vec{r},\vec{\Omega}';\vec{r}_0,\vec{\Omega}_0)\,f(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},-\vec{\Omega};\vec{r}_1,-\vec{\Omega}_1)\,d\Omega\,d\Omega' \tag{11}$$

$$= \iint\psi(\vec{r},\vec{\Omega};\vec{r}_0,\vec{\Omega}_0)\,f(\vec{\Omega}\cdot\vec{\Omega}')\,\psi(\vec{r},-\vec{\Omega}';\vec{r}_1,-\vec{\Omega}_1)\,d\Omega\,d\Omega'$$

Hence (10) becomes:

$$\psi(\vec{r}_1, \vec{\Omega}_1; \vec{r}_o, \vec{\Omega}_o) = \psi(\vec{r}_o, -\vec{\Omega}_o; \vec{r}_1, -\vec{\Omega}_1) \tag{12}$$

which is the Reciprocity Theorem.

A continuity equation in a more restricted sense is obtained by integrating the transport equation (3) over $\vec{\Omega}$. This gives

$$v \operatorname{div} \vec{j} + v \sigma \rho = q_o + v \sigma c \rho \tag{13}$$

or

$$\operatorname{div} \vec{j}(\vec{r}) = \frac{q_o(\vec{r})}{v} + \sigma(\vec{r}) \rho(\vec{r}) \left[c(\vec{r}) - 1\right] \tag{14}$$

where

$$q_o(\vec{r}) = \int q(\vec{r}, \vec{\Omega}) \, d\Omega \tag{15}$$

(14) is usually referred to as the <u>Continuity Equation</u>.

A direct series (Neumann) solution of (3) is obtained by successive approximation. The zeroth approximation is found by dropping the integral term. (3) then becomes identical with (5-3). The solution given by (5-11) is:

$$\psi_o(\vec{r}, \vec{\Omega}) = \frac{1}{v} \int_0^\infty q(\vec{r} - R\vec{\Omega}, \vec{\Omega}) \; e^{-\alpha(\vec{r}, \vec{r} - R\vec{\Omega})} \, dR \tag{16}$$

For the n'th approximation for ψ we substitute the n-1'th approximation for ψ in the integral term. The right hand side of (3) then becomes a known source of modified strength and distribution. Thus if we define q_n by:

$$q_n(\vec{r}, \vec{\Omega}) = q(\vec{r}, \vec{\Omega}) + c(\vec{r}) \, v \sigma(\vec{r}) \int \psi_{n-1}(\vec{r}, \vec{\Omega}') \, f(\vec{\Omega}, \vec{\Omega}') \, d\Omega' \tag{17}$$

(5-11) gives for the n'th approximation

$$\psi_n(\vec{r}, \vec{\Omega}) = \frac{1}{v} \int_0^\infty q_n(\vec{r} - R\vec{\Omega}, \vec{\Omega}) \, e^{-\alpha(\vec{r}, \vec{r} - R\vec{\Omega})} \, dR \tag{18}$$

The successive approximations given by (18) differ merely by the inclusion of successively more collisions. Thus ψ_0 is the angular density of neutrons which have suffered no collisions, ψ_1 includes those which have suffered one collision, ψ_2 includes two collisions, etc. From this it is apparent that only for c small or for bodies of small size will the successive terms converge rapidly. For steady state problems this Neuman solution negates the whole use of the Transport Equation. The whole importance of the latter is that it enables us to solve such problems in a compact form rather than by considering all the collisions that neutrons coming from the source experience.

For time dependent problems the transport equation must be modified (in analogy with (2-11) by a term expressing the time rate of change of the angular density at the point $\vec{r}$. Thus (3) must be changed to

$$\frac{\partial \Psi(\vec{r},\vec{\Omega},t)}{\partial t} + v \operatorname{div} \vec{\Omega}\, \Psi(\vec{r},\vec{\Omega},t) + v\,\sigma(\vec{r})\, \Psi(\vec{r},\vec{\Omega},t)$$
$$= q(\vec{r},\vec{\Omega},t) + c(\vec{r})\, v\,\sigma(\vec{r}) \int \Psi(\vec{r},\vec{\Omega}',t)\, f(\vec{\Omega},\vec{\Omega}')\, d\Omega' \tag{19}$$

To solve (19) with the condition

$$\Psi = 0 \qquad t \leqslant 0 \tag{20}$$

it is convenient to introduce the one sided Laplace Transform defined by

$$\psi(\vec{r},\vec{\Omega},s) = \int_0^\infty \Psi(\vec{r},\vec{\Omega},t)\, e^{-st}\, dt \tag{21}$$

Multiplying (19) by e^{-st} and integrating from zero to infinity gives on using (20)

$$s\,\psi(\vec{r},\vec{\Omega},s) + v \operatorname{div} \vec{\Omega}\, \psi(\vec{r},\vec{\Omega},s) + v\,\sigma(\vec{r})\, \psi(\vec{r},\vec{\Omega},s)$$
$$= \int_0^\infty q(\vec{r},\vec{\Omega},t)\, e^{-st}\, dt + v\,\sigma\, c(\vec{r}) \int \psi(\vec{r},\vec{\Omega}',s)\, f\, d\Omega' \tag{22}$$

If q describes a pulse at t = 0, i.e.

$$q(\vec{r}, \vec{\Omega}, t) = q(\vec{r}, \vec{\Omega})\,\delta(t) \qquad (23)$$

(22) becomes:

$$v\,\mathrm{div}\,\vec{\Omega}\,\psi(\vec{r},\vec{\Omega},s) + (v\sigma + s)\,\psi(\vec{r},\vec{\Omega},s) = q(\vec{r},\vec{\Omega}) + vc\sigma\int \psi(\vec{r},\vec{\Omega}',s)\, f\, d\Omega' \qquad (24)$$

Defining

$$\sigma' = \sigma + s/v$$
$$c' = \frac{c}{1 + s/\sigma v} \qquad (25)$$

brings (24) into the form

$$v\,\mathrm{div}\,(\vec{\Omega}\,\psi) + v\sigma'\psi = q + vc'\,\sigma'\int \psi\; f\, d\Omega' \qquad (26)$$

which is the same as (3) with modified σ and c. In particular if σ be constant we can introduce units of length and time such that σ' and $v\sigma'$ are unity. This reduces (26) to

$$\mathrm{div}\,\vec{\Omega}\,\psi + \psi = q + c'\int \psi\; f\, d\Omega' \qquad (27)$$

The only difference between (27) and the corresponding steady state equation is the change of c to c'. Thus we see that one interpretation of the stationary $\psi(\vec{r}, \vec{\Omega})$ is as the Laplace Transform of the time dependent angular density. Alternatively the steady state solution may be obtained from the time dependent one. If $\Psi_\tau(\vec{r}, \vec{\Omega}, t)$ is the solution of (19) with the source (23) pulsed at τ, the solution of (3) is

$$\psi(\vec{r}, \vec{\Omega}) = \int_{-\infty}^{t} \Psi_\tau(\vec{r}, \vec{\Omega}, t)\, d\tau \qquad (28)$$

12. Integral Equations

Let us first consider the case of isotropic scattering i.e.

$$f(\vec{\Omega}\cdot\vec{\Omega}') = \frac{1}{4\pi} \tag{1}$$

In this case it is sufficient to assume isotropic sources. Anisotropic sources may be treated by dividing the neutrons at a point $\vec{r}$ into those coming directly from the source and those which have suffered a collision first. With assumption (1) those neutrons which have suffered one collision will be describable as an isotropic source.

With these simplifications (11-3) becomes

$$v \, \mathrm{div}_{\vec{\Omega}} \, \psi(\vec{r},\vec{\Omega}) + v\,\sigma(\vec{r})\,\psi(\vec{r},\vec{\Omega}) = \frac{q_o(\vec{r}) + cv\sigma\rho(\vec{r})}{4\pi} \tag{2}$$

In accordance with the idea discussed in section 11 we shall regard the right hand side of (2) as the source occurring in equation (5-3). The solution (5-11) of this equation then tells us that

$$\psi(\vec{r},\vec{\Omega}) = \frac{1}{4\pi}\int \left\{ \frac{q_o(\vec{r}-R\vec{\Omega})}{v} + c(\vec{r}-R\vec{\Omega})\;\rho(\vec{r}-R\vec{\Omega})\; e^{-\alpha(\vec{r},\vec{r}-R\vec{\Omega})} \right\} dR \tag{3}$$

while (5-12) tells us that

$$\rho(\vec{r}) = \frac{1}{4\pi}\int \left\{ \frac{q_o(\vec{r}\,')}{v} + c(\vec{r}\,')\;\rho(\vec{r}\,') \right\} \frac{e^{-\alpha(\vec{r},\vec{r}\,')}}{|r-r'|^2}\; d\vec{r}\,' \tag{4}$$

(4) is an integral equation for the density ρ. It is considerably simpler than the integro-differential equation (2) as it only involves a function of the position coordinate r. If we solve (4) the angular density ψ may be obtained by insertion of the resulting $\rho(\vec{r})$ in (3).

This procedure for the solution of the transport equation is readily generalized to the case of scattering functions f which are, or can be approximated by, polynomials in $\vec{\Omega}\cdot\vec{\Omega}'$. (Scattering laws, such as the Rutherford law, which are singular are thus excluded). If f be such a polynomial of degree n in $\vec{\Omega}\cdot\vec{\Omega}'$ the transport equation will have as effective sources terms involving angular moments of ψ up to the n'th order. Using these terms as the source in (5-11) gives, in analogy with (3), the angular density in terms of these moments. Multiplying this equation by various powers of the components of $\vec{\Omega}$ and integrations over $\vec{\Omega}'$ gives as the analog of (4) a set of integral equations for the moments involved. After solving these equations one can go back and calculate the angular density.

IV. Uniform Infinite Medium with Isotropic Scattering

For this case in which c and σ are independent of position the formulae can be simplified by taking σ and v to be unity (i.e., we measure distance in units of the mean free path ℓ and time in units of ℓ/v).

13. Source Free Solution

In these new units equation (12-4) gives as the integral equation for the neutron density in the absence of sources

$$\rho(\vec{r}) = c\int \rho(\vec{r}\,')\, \frac{e^{-|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|^2}\, d\vec{r}\,' \tag{1}$$

Consider solutions of the form

$$\rho(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \tag{2}$$

with

$$\vec{k} = k\vec{u} \tag{3}$$

Here k is a complex number and $\vec{u}$ a unit vector. These are to be determined so that (2) is a solution of (1). Inserting (2) in (1) gives

$$e^{i\vec{k}\cdot\vec{r}} = \frac{c\,\tan^{-1}k}{k}\, e^{i\vec{k}\cdot\vec{r}} \tag{4}$$

Hence (2) is a solution if $k = k_o$ where k_o is a solution of

$$1 - c \frac{\tan^{-1} k_o}{k_o} = 0 \tag{5}$$

Thus

$$\rho_{\vec{u}}(\vec{r}) = e^{ik_o \vec{u} \cdot \vec{r}} \tag{6}$$

is a solution for arbitrary u. A more general solution of (1) is obtained by adding solutions of the form (6). Thus:

$$\rho(\vec{r}) = \int f(\vec{u}) \, e^{ik_o \vec{u} \cdot \vec{r}} \, du \tag{7}$$

where $f(\vec{u})$ is an arbitrary function.

With the restriction that $\rho(r)e^{-r}$ is to vanish for r going to infinity (7) constitutes the most general solution of (1). This may be shown by a simple generalization of a proof given by Titchmarsh*. Titchmarsh treats a rather broad class of one dimensional, singular, displacement integral equations. Here we will restrict ourselves to stating the proof for the one-dimensional form of (1) and then sketching briefly the course of a proof of the complete generality of the solution (7).

Consider the one dimensional integral equation:

$$\rho(z) = \frac{c}{2} \int_{-\infty}^{\infty} \rho(z') \, E\,(|z-z'|) \, dz' \tag{8}$$

Try as a solution:

$$\rho_o(z) = e^{ikz} \tag{9}$$

Inserting in (8) yields:

$$\left(1 - \frac{c \tan^{-1} k}{k}\right) e^{ikz} = 0 \tag{10}$$

Thus

$$\rho_o(z) = e^{ik_o z} \tag{11}$$

---

*E. C. Titchmarsh, "Theory of Fourier Integrals" Sec. 11-2.

solves (8) provided k_o satisfies

$$\left(1 - \frac{c \tan^{-1} k_o}{k_o}\right) = 0 \tag{12}$$

There are two solutions of (12) which may be denoted by k_o and $-k_o$.
A general solution of (8) is then

$$\rho(z) = Ae^{ik_o z} + Be^{-ik_o z} \tag{13}$$

The theorem to be proven is that all solutions of (8) such that $\rho(z)e^{-|z|}$ is bounded by $Me^{-\epsilon|z|}$ as $|z| \to \infty$ are included in (13) for arbitrarily small $\epsilon > 0$. To prove this the following lemma is needed.

<u>Lemma</u>:

Let $\phi(k)$ be regular in the strip (where $k = \sigma + it$) $a_1 \leq t \leq 1$ and $\phi(\sigma + it)$ be absolutely integrable over σ from $-\infty$ to ∞ in this strip. Let $\psi(k)$ have the same properties in the strip $-1 \leq t \leq b_1$ where $b_1 < a_1$

Let

$$\int_{ia-\infty}^{ia+\infty} \phi(k)\, e^{-izk}\, dk + \int_{ib-\infty}^{ib+\infty} \psi(k)\, e^{-izk}\, dk = 0 \tag{14}$$

for all z, where a and b are in the regularity strips of ϕ and ψ, respectively.

Multiplying (14) by e^{izw} where $w = \xi + i\eta$ and $a < \eta < 1$, integrating with respect to z from o to ∞, and interchanging the orders of integration gives:

$$\int_{ia-\infty}^{ia+\infty} \frac{\phi(k)}{k-w}\, dk + \int_{ib-\infty}^{ib+\infty} \frac{\psi(k)}{k-w}\, dk = 0 \qquad (a < \eta < 1) \tag{15}$$

By Cauchy's Theorem:

$$\left(\int_{ia-\infty}^{ia+\infty} - \int_{ib-\infty}^{ib+\infty}\right) \frac{\phi(k)}{k-w}\, dk = -2\pi i\, \phi(w) \tag{16}$$

Subtracting (16) from (15) yields:

$$\int_{ib-\infty}^{ib+\infty} \frac{\phi(k)}{k-w}\, dk + \int_{ib-\infty}^{ib+\infty} \frac{\psi(k)}{k-w}\, dk = -2\pi i\, \phi(w) \qquad (a < \eta < 1) \tag{17}$$

The left hand side of (17) is analytic for $b < \eta < 1$ and hence provides the continuation of $\phi(w)$ throughout this strip.

Similarly one can show that

$$\int_{ia-\infty}^{ia+\infty} \frac{\phi(k)}{k-w}\, dk + \int_{ia-\infty}^{ia+\infty} \frac{\psi(k)}{k-w}\, dk = +2\pi i\, \psi(w) \qquad (-1 < \eta < b) \tag{18}$$

which gives the analytic continuation of $\psi(w)$ over $-1 < \eta < a$. If $b < \eta < a$, the left hand sides of (17) and (18) are equal and so in this region

$$\phi(w) = -\psi(w) \tag{19}$$

and $\phi(w)$ is analytic in the strip.

With the same conditions it can be shown that $\phi(w)$ tends to zero uniformly in this strip as $|\xi| \to \infty$

The completeness theorem is now readily demonstrated. Let

$$\begin{aligned} K(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{c}{2} E_1(|z|)\, e^{ikz}\, dz \\ &= \frac{1}{\sqrt{2\pi}}\, \frac{c \tan^{-1} k}{k} \end{aligned} \tag{20}$$

This is regular for $-1 < t < 1$.

Let

$$\Phi_+(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \rho(z)\, e^{ikz}\, dz \tag{21}$$

$$\Phi_-(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \rho(z)\, e^{ikz}\, dz \tag{22}$$

which by the conditions on the solution $\rho(z)$ are regular in the regions $1 - \epsilon < t$ and $t < -(1 - \epsilon)$, respectively. Then, if $1 - \epsilon < a < 1$, $-1 < b < -(1 - \epsilon)$

$$\frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} \Phi_+(k)\, e^{-izk}\, dk = \begin{cases} \rho(z) & z > 0 \\ 0 & z < 0 \end{cases} \tag{23}$$

$$\frac{1}{\sqrt{2\pi}} \int_{ib-\infty}^{ib+\infty} \Phi_-(k)\, e^{-izk}\, dk = \begin{cases} 0 & z > 0 \\ \rho(z) & z < 0 \end{cases} \tag{24}$$

By the Convolution Theorem:

$$\frac{c}{2} \int_0^\infty \rho(z')\, E_1(|z - z'|)\, dz' = \int_{ia-\infty}^{ia+\infty} \Phi_+(k)\, K(k)\, e^{-izk}\, dk \tag{25}$$

$$\frac{c}{2} \int_{-\infty}^0 \rho(z')\, E_1(|z - z'|)\, dz' = \int_{ib-\infty}^{ib+\infty} \Phi_-(k)\, K(k)\, e^{-izk}\, dk \tag{26}$$

Inserting (23) - (26) in (8) gives:

$$\int_{ia-\infty}^{ia+\infty} \Phi_+(k) \left\{1 - \sqrt{2\pi}\, K(k)\right\} e^{-izk}\, dk \tag{27}$$

$$+ \int_{ib-\infty}^{ib+\infty} \Phi_-(k) \left\{1 - \sqrt{2\pi}\, K(k)\right\} e^{-izk}\, dk = 0$$

Using the lemma one sees from (27) that

$$\Phi_+(k) \left\{1 - \sqrt{2\pi}\, K(k)\right\} = - \Phi_-(k) \left\{1 - \sqrt{2\pi}\, K(k)\right\} \quad dk \tag{28}$$

throughout the strip $b < t < a$, and both sides are analytic there. Hence $\Phi_+(k) = - \Phi_-(k)$ and $\Phi_+(k)$ is analytic except possibly for simple poles at the two zeros k_o and $-k_o$ of

$$1 - \sqrt{2\pi}\, K(k) = 1 - \frac{c \tan^{-1} k}{k} \tag{29}$$

Adding (23) to (24) and using the relation between Φ_+ and Φ_- gives:

$$\rho(z) = \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} \Phi_+(k)\, e^{-izk}\, dk - \frac{1}{\sqrt{2\pi}} \int_{ib-\infty}^{ib+\infty} \Phi_+(k)\, e^{-izk}\, dk \tag{30}$$

Since $\Phi_+(k)$ vanishes at infinity in the strip $b < t < a$, the integrals in (30) may be evaluated by residues giving

$$\rho(z) = (\text{const.})\, e^{ik_oz} + (\text{const.})\, e^{-ik_oz} \tag{31}$$

which is what was to be shown.

It can be seen that a corresponding proof for the generality of the solution (7) of (1) is readily constructed. Instead of Fourier Transforms of one complex variable k, one must take Transforms in terms of three complex variables k_x, k_y, k_z. Decomposing $\rho(\vec{r})$ into eight functions which are equal to it in one octant and zero elsewhere one can define the eight corresponding transforms, each with their appropriate regions of analyticity in complex k space. Paralleling the one dimensional proof the various transforms can be shown equal up to sign and analytic except for simple poles at the zeros of $1 - \frac{c}{2} \frac{\tan^{-1} k}{k}$ where $\vec{k} = k\vec{\Omega}$.

Expressing $\rho(\vec{r})$ in terms of integrals over the Fourier Transforms and changing to polar coordinates in k space makes it again possible to evaluate $\rho(\vec{r})$ by means of residues. This leaves $\rho(\vec{r})$ in the form (7). Hence (7) is the general solution of (1) consistent with the condition that $\rho(\vec{r})\, e^{-r}$ be bounded by $Me^{-\epsilon r}$ for $r \to \infty$ with some arbitrarily small $\epsilon > 0$.

From (7) it is very easy to obtain many interesting properties of the infinite medium solutions. Thus on differentiating we obtain

$$\begin{aligned} \nabla^2\rho(\vec{r}) &= \int f(\vec{u})\ \nabla^2\, e^{ik_o \vec{u}\cdot\vec{r}}\, du \\ &= -\,k_o^{\,2}\int f(\vec{u})\ e^{ik_o\vec{u}\cdot\vec{r}}\, du \end{aligned} \tag{32}$$

or

$$(\nabla^2 + k_o^{\,2})\ \rho(\vec{r}) = 0 \tag{33}$$

i.e. $\rho(\vec{r})$ satisfies the wave equation with wave number given by (5).

Equation (33) is particularly helpful in finding the infinite medium solutions corresponding to special symmetries. Thus on the assumption of spherical symmetry (33) becomes:

$$\frac{1}{r}\frac{d^2}{dr^2}\,(r\rho(r)) + k_o^{\,2}\rho(r) = 0 \tag{34}$$

with the unique solution for all r:

$$\rho(r) = (\text{constant})\ \frac{\sin k_o r}{r} \tag{35}$$

In case of plane symmetry where the only variation is in the z direction (33) simplifies to

$$\frac{d^2}{dz^2}\ \rho(z) + k_o^{\,2}\rho(z) = 0 \tag{36}$$

with the general solution

$$\rho(z) = c_1 \sin k_o z + c_2 \cos k_o z \tag{37}$$

Returning to the solution (7) we see that under the conditions of this section (12-3) gives for the angular density

$$\psi(\vec{r},\vec{\Omega}) = \frac{c}{4\pi}\int_0^\infty \rho(\vec{r}-R\vec{\Omega})\, e^{-R}\, dR \tag{38}$$

Inserting the solution (7) gives

$$\psi = \frac{c}{4\pi}\iint dR\, du\, f(\vec{u})\, e^{-R}\, e^{ik_o\vec{u}\cdot(\vec{r}-R\vec{\Omega})} \tag{39}$$

On carrying out the R integration

$$\psi(\vec{r},\vec{\Omega}) = \frac{c}{4\pi}\int \frac{f(\vec{u})\; e^{ik_o\vec{u}\cdot\vec{r}}}{1 \quad ik_o\vec{u}\cdot\vec{\Omega}}\, du \tag{40}$$

Let us define the n'th angular moment $M_n^{n_i}$ by

$$M_n^{n_i} = \int \left(\prod_{i=1}^{3} \Omega_i^{n_i}\right) \psi(\vec{r},\vec{\Omega})\, d\Omega \tag{41}$$

where

$$\sum_{i=1}^{3} n_i = n \tag{42}$$

Then with

$$\phi_n(\vec{u}) = \frac{c}{4\pi}\int \frac{\left(\prod_{i=1}^{3} \Omega_i^{n_i}\right) d\Omega}{1 \quad ik_o\vec{u}\cdot\vec{\Omega}} \tag{43}$$

we obtain on multiplying (40) by $\prod_i \Omega_i^{n_i}$ and integrating over $\vec{\Omega}$:

$$M_n = \int f(\vec{u})\, e^{ik_o\vec{u}\cdot\vec{r}}\, \phi_n(\vec{u})\, du \tag{44}$$

For $n = 0$, $\phi_n = 1$ and (44) gives

$$M_o = \int \psi(\vec{r},\vec{\Omega})\, d\Omega = \rho = \int f(\vec{u})\, e^{ik_o\vec{u}\cdot\vec{r}}\, du \tag{45}$$

which is just (7).

For $n = 1$, $\phi_1^{\,i}$ and $M_1^{\,i}$ are components of vectors. ($\vec{M}_1$ is just the current in the present units.)

$$\phi_1^{\,i} = \frac{c}{4\pi}\int \frac{\Omega^i d\,\Omega}{1 + ik_o\vec{u}\cdot\vec{\Omega}} \tag{46}$$

Since $\vec{u}$ is the only vector in (46) symmetry tells us

$$\phi_1^{\,i} = u^i\,\psi(u^2) = u^i\,\psi(1) \tag{47}$$

as

$$\vec{u}\cdot\vec{u} = u^2 = 1 \tag{48}$$

Taking the scalar product of the vector $\vec{\phi}_1$, with $\vec{u}$ gives on remembering (48):

$$\psi(1) = \frac{c}{4\pi}\int \frac{\vec{u}\cdot\vec{\Omega}\,d\,\Omega}{1 + ik_o\vec{u}\cdot\vec{\Omega}} = \frac{c - 1}{ik_o} \tag{49}$$

Hence

$$\phi_1^{\,i} = \frac{c - 1}{ik_o}\,u^i \tag{50}$$

and

$$M_1^{\,i} = \frac{c - 1}{ik_o}\int f(\vec{u})\,e^{ik_o(\vec{u}\cdot\vec{r})}\,u^i\,du \tag{51}$$

But since

$$\frac{1}{ik_o}\,\frac{\partial}{\partial x_i}\int f(\vec{u})\,e^{ik_o\,\vec{u}\cdot\vec{r}} = \int f(\vec{u})\,e^{ik_o\vec{u}\cdot\vec{r}}\,u^i\,du \tag{52}$$

$$M_1^{\,i} = \frac{(c-1)}{k_o^{\,2}}\,\frac{\partial}{\partial x_i}\int f(\vec{u})\,e^{ik_o\vec{u}\cdot\vec{r}}\,du = -\frac{(c-1)}{k_o^{\,2}}\,\frac{\partial}{\partial x_i}\,\rho(\vec{r}) \tag{53}$$

or

$$\vec{j} = -\,D\cdot\text{grad}\,\rho \tag{54}$$

where

$$D = \frac{c-1}{k_o^{\,2}} \tag{55}$$

Incidentally it might be noted that elementary diffusion theory is based on the assumption that (54) does not only hold in a source free infinite medium but that it has generally validity and that furthermore D = 1/3. The range of validity of this approximation will be discussed later

For n = 2, M_2 and ϕ_2 are symmetric second rank tensors.

$$\phi_2^{ij} = \frac{c}{4\pi} \int \frac{\Omega^i \Omega^j}{1 + ik_o \vec{u} \cdot \vec{\Omega}} \, d\Omega \tag{56}$$

Again by symmetry we see

$$\phi_2^{ij} = u^i u^j \chi_1(u^2) + \delta^{ij} \chi_2(u^2) = u^i u^j \chi_1(1) + \delta^{ij} \chi_2(1) \tag{57}$$

Taking the trace of ϕ_2 we obtain

$$\mathrm{Tr}\,\phi_2 = \phi_2^{ii} = \chi_1 + 3\,\chi_2 \tag{58}$$

But from (56)

$$\mathrm{Tr}\;\phi_2 = \frac{c}{4\pi} \int \frac{d\Omega}{1 + ik_o \vec{u} \cdot \vec{\Omega}} = 1 \tag{59}$$

Hence

$$\chi_1 + 3\,\chi_2 = 1 \tag{60}$$

Multiplying ϕ_2^{ij} by $u^i u^j$ and summing gives

$$u^i u^j \;\phi_2^{ij} = \chi_1 + \chi_2 = \frac{c}{4\pi} \int \frac{(\vec{u} \cdot \vec{\Omega})^2 d\Omega}{1 + ik_o \vec{u} \cdot \vec{\Omega}} \tag{61}$$

or

$$\chi_1 + \chi_2 = \frac{c-1}{k_o^2} = D \tag{62}$$

Solving (60) and (62) yields

$$\chi_2 = \frac{(1-D)}{2} \tag{63}$$

$$\chi_1 = \frac{(3D-1)}{2} \tag{64}$$

Inserting (57), (63) and (64) in (44) gives on using (52)

$$M_2^{ij} = -\frac{1}{k_o^2}\,\frac{(3D-1)}{2}\,\frac{\partial^2 \rho}{\partial x_i \partial x_j} + \frac{(1-D)}{2}\,\delta^{ij}\rho \qquad (65)^*$$

In plane problems where the only spatial variations are in the direction of the vector $\vec{n}$

$$n_i n_j M_2^{ij} = -\frac{1}{k_o^2}\,\frac{(3D-1)}{2}\,\frac{\partial^2 \rho}{\partial n^2} + \frac{(1-D)}{2}\,\rho \tag{66}$$

or on using (36)

$$\frac{n_i n_j M_2^{ij}}{\rho} = \frac{\int (\vec{n}\cdot\vec{\Omega})^2 \psi(\vec{r},\vec{\Omega})\,d\Omega}{\int \psi(\vec{r},\vec{\Omega})\,d\Omega} = D \tag{67}$$

It should be noted that by methods similar to those used here one can express the n'th moment in terms of the first n derivatives of the density.

Since the roots of (5) are of decisive importance for further developments we will now examine them in detail. We will also consider various significant functions of these roots.

For $c > 1$ k_o is real. If $c < 1$ it is imaginary. In this latter case we will set $k_o = i\,\chi_o$. Then

$$c = \frac{\chi_o}{\tanh^{-1} \chi_o} \tag{68}$$

---

*The results (54) and (65) have been obtained by P. R. Wallace, Canadian Journal of Research 26A, 114, (1948).

In particular

$$\left.\begin{array}{lll} c = 0 & : & \varkappa_0 = 1 \\ c = 1 & : & k_0 = \varkappa_0 = 0 \\ c \to \infty & : & k_0 \to \frac{\pi}{2} c \end{array}\right\} \qquad (69)$$

Graphs of $\varkappa_0$ and k_0 are given in Figures 16 and 17. The diffusion length is defined as $1/\varkappa_0$. Fig. 16 shows that with decreasing c the diffusion length moves very rapidly to unity, which in our units corresponds to the mean free path. Even for $c = 0.6$ we find $\varkappa_0 = 0.907$. This shows that scattering must predominate to a very great extent before the diffusion length is appreciably larger than the mean free path.

The following expansions represent k_0 and $\varkappa_0$ in a wide range of c.

$c \ll 1$

$$\varkappa_0 = 1 - 2e^{-\frac{2}{c}}\Big(1 + \frac{4-c}{c} e^{-\frac{2}{c}} + \frac{24-12c+c^2}{c^2} e^{-\frac{4}{c}} + \frac{512-384c+72c^2-3c^3}{3c^3} e^{-\frac{6}{c}} + \ldots\Big) \qquad (70)$$

$$\varkappa_0^2 = 1 - 4e^{-\frac{2}{c}}\left\{1 + \frac{4-2c}{c} e^{-\frac{2}{c}} + \frac{24-20c+3c^2}{c^2} e^{-\frac{4}{c}} + \frac{512-576c+168c^2-12c^3}{3c^3} e^{-\frac{6}{c}} + \ldots\right\} \qquad (71)$$

and

$$\frac{1}{\varkappa_0^2} = 1 + 4e^{-\frac{2}{c}}\left\{1 + \frac{4+2c}{c} e^{-\frac{2}{c}} + \frac{24+12c+3c^2}{c^2} e^{-\frac{4}{c}} + \frac{512+192c+72c^2+12c^3}{3c^3} e^{-\frac{6}{c}} + \ldots\right\} \qquad (72)$$

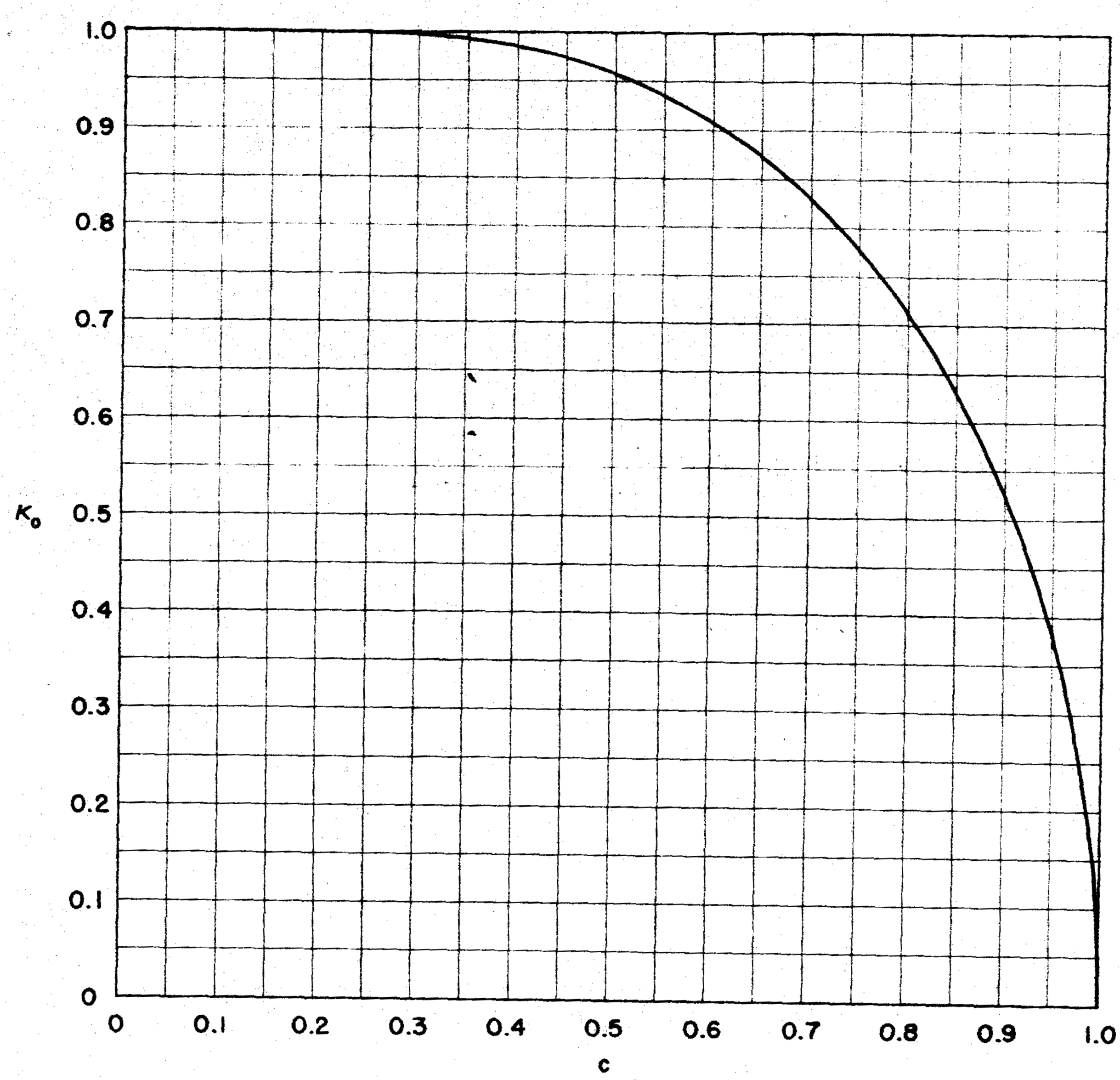


Fig. 16

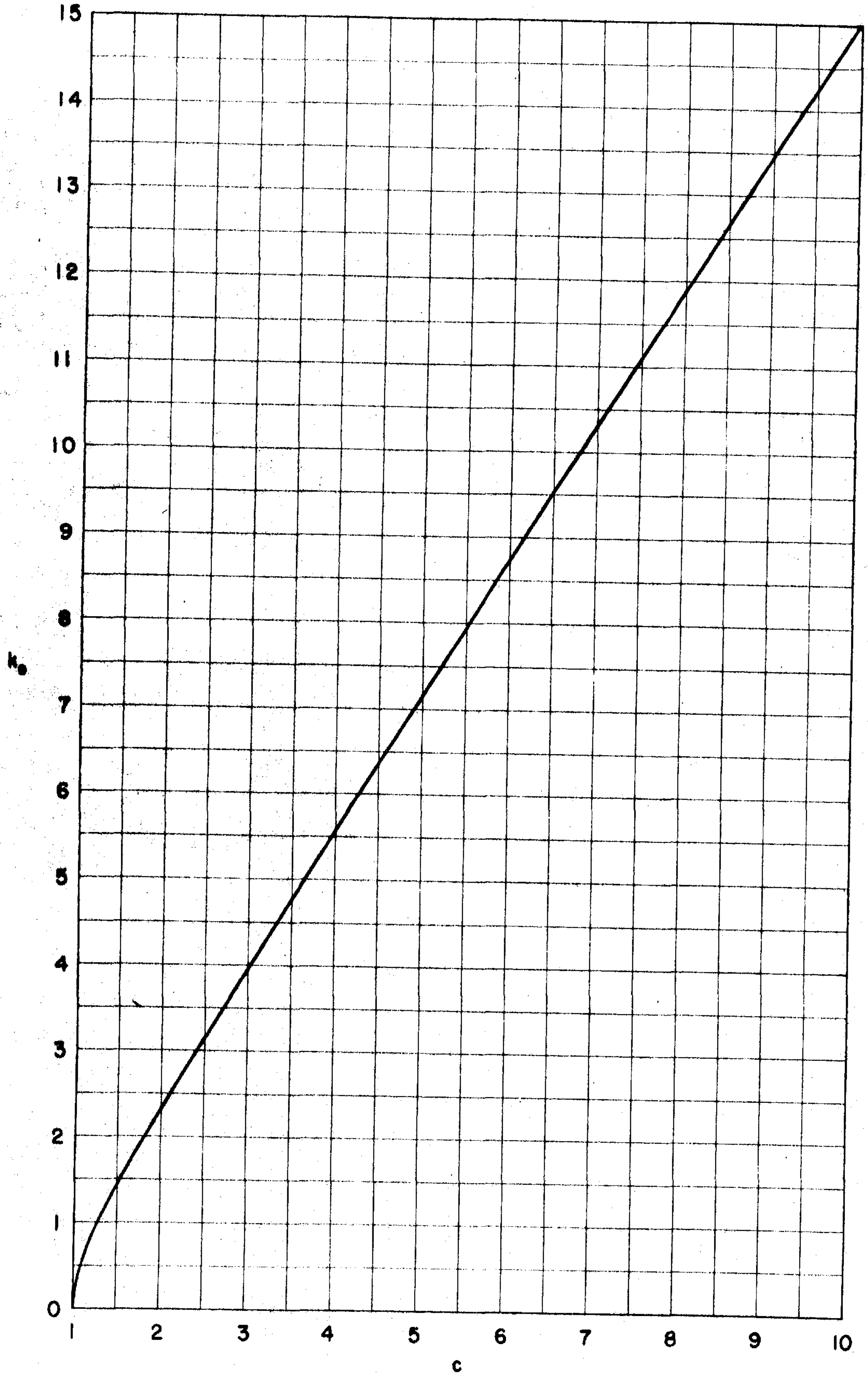


Fig. 17

$|c - 1| << 1$

$$\frac{\mathcal{K}_o}{\sqrt{3(1 - c)}} = \frac{k_o}{\sqrt{3(c - 1)}} \tag{73}$$

$$= \left\{1 - \frac{2}{5}(1 - c) - \frac{12}{175}(1-c)^2 - \frac{2}{125}(1-c)^3 + \frac{166}{67375}(1-c)^4 \ldots\right\}$$

$$= \left\{1 - 0.4(1-c) - 0.0685714(1-c)^2 - 0.016(1-c)^3 + 0.0024638(1-c)^4 \ldots\right\}$$

$$k_o^2 = -\mathcal{K}_o^2 = 3(c-1)\left\{1 - \frac{4}{5}(1-c) + \frac{4}{175}(1-c)^2\right.$$

$$\left. + \frac{4}{175}(1-c)^3 + \frac{7556}{336875}(1-c)^4 + \frac{471844}{21896875}(1-c)^5 + \ldots\right\} \tag{74}$$

$$= 3(c-1)\left\{1 - 0.8(1-c) + 0.022857143(1-c)^2\right.$$

$$+ 0.022857143(1-c)^3 + 0.02242968(1-c)^4$$

$$\left. + 0.02154846(1-c)^5 + \ldots\right\}$$

and

$$\frac{1}{k_o^2} = \frac{-1}{\mathcal{K}_o^2} =$$

$$\frac{1}{3(c-1)}\left\{1 + \frac{4}{5}(1-c) + \frac{108}{175}(1-c)^2 + \frac{396}{875}(1-c)^3\right. \tag{75}$$

$$\left. + \frac{828}{2695}(1-c)^4 + \frac{568836}{3128125}(1-c)^5 + \ldots\right\}$$

$$= \frac{1}{3(c-1)}\left\{1 + 0.8(1-c) + 0.6171429(1-c)^2 + 0.4525714(1-c)^3\right.$$

$$\left. + 0.3072356(1-c)^4 + 0.1818457(1-c)^5 + \ldots\right\}$$

c >> 1:

$$k_o = \frac{\pi}{2} c \left\{ 1 - \frac{4}{\pi^2} \left(\frac{1}{c}\right) - \frac{16}{\pi^4} \left(\frac{1}{c}\right)^2 - \left(\frac{384 - 16\pi^2}{3\pi^6}\right) \left(\frac{1}{c}\right)^3 \right.$$

$$\left. - \left(\frac{3840 - 256\pi^2}{3\pi^8}\right) \left(\frac{1}{c}\right)^4 \dots \right\} \quad (76)$$

$$= \frac{\pi}{2} c \left\{ 1 - \frac{0.4052847}{c} - \frac{0.1642557}{c^2} - \frac{0.0783888}{c^3} - \frac{0.0461393}{c^4} - \dots \right\}$$

$$k_o^2 = \frac{\pi^2 c^2}{4} \left\{ 1 - \frac{8}{\pi^2} \left(\frac{1}{c}\right) - \frac{16}{\pi^4} \left(\frac{1}{c}\right)^2 - \left(\frac{384 - 32\pi^2}{3\pi^6}\right) \left(\frac{1}{c}\right)^3 \right.$$

$$\left. - \left(\frac{3840 - 384\pi^2}{3\pi^8}\right) \left(\frac{1}{c}\right)^4 \dots \right\} \quad (77)$$

$$\frac{1}{k_o^2} = \frac{4}{\pi^2 c^2} \left\{ 1 + \frac{8}{\pi^2} \left(\frac{1}{c}\right) \quad \frac{80}{\pi^4} \left(\frac{1}{c}\right)^2 + \left(\frac{2688 - 32\pi^2}{3\pi^6}\right) \left(\frac{1}{c}\right)^3 \right.$$

$$\left. \left(\frac{32256 - 896\pi^2}{3\pi^8}\right) \left(\frac{1}{c}\right)^4 \dots \right\} \quad (78)$$

Future applications will require knowledge of $\frac{dk_o^2}{dc}$. Differentiating (5) we obtain:

$$\frac{dk_o^2}{dc} = \frac{2k_o^2(1 + k_o^2)}{c\left[1 - c \quad k_o^2\right]} \quad (79)$$

Differentiating the above series expansions for k_o^2 gives:

c << 1:

$$\frac{dk_o^2}{dc} = \frac{8}{c^2} e^{-2/c} \left\{ 1 + \frac{8 - 6c}{c} e^{-2/c} + \frac{72 - 84c + 19c^2}{c^2} e^{-4/c} \right.$$

$$\left. \frac{2048 - 3072c + 1248c^2 - 132c^3}{3c^3} e^{-6/c} + \dots \right\} \quad (80)$$

$|c - 1| << 1$:

$$\frac{dk_o^2}{dc} = 3 \left\{ 1 - \frac{8}{5}(1-c) + \frac{12}{175}(1-c)^2 + \frac{16}{175}(1-c)^3 + \frac{7556}{67375}(1-c)^4 + \frac{2831064}{21896875}(1-c)^5 + \ldots \right\} \tag{81}$$

$c >> 1$:

$$\frac{dk_o^2}{dc} = \frac{\pi^2}{2} c \left\{ 1 - \frac{4}{\pi^2 c} + \frac{0.011181844}{c^3} + \frac{0.0017622}{c^4} + \ldots \right\} \tag{82}$$

We have seen that D defined by

$$D = \frac{c-1}{k_o^2} = \frac{1-c}{\kappa_o^2} \tag{83}$$

occurs in many fundamental ways. From the previous results for k_o we find

$$\left.\begin{array}{lll} c = 0 & : & D = 1 \\ c = 1 & : & D = 1/3 \\ c = \infty & : & D = 0 \end{array}\right\} \tag{84}$$

Expansions for D are:

$c << 1$

$$D = (1-c) \left\{ 1 + 4e^{-\frac{2}{c}} + 8\,\frac{c+2}{c}\, d^{-\frac{4}{c}} + 12\,\frac{8+4c+c^2}{c^2}\, e^{-\frac{6}{c}} + 16\,\frac{128+48c+18c^2+3c^3}{3c^3}\, e^{-\frac{8}{c}} + \ldots \right\} \tag{85}$$

$|c - 1| << 1$

$$D = \frac{1}{3} \left\{ 1 + \frac{4}{5}(1-c) + \frac{108}{175}(1-c)^2 + \frac{396}{875}(1-c)^3 + \frac{828}{2695}(1-c)^4 + \frac{568836}{3128125}(1-c)^5 + \ldots \right\} \tag{86}$$

It may be noted that (73) is the expansion of $\frac{1}{\sqrt{3D}}$ in this region.

c >> 1

$$D = \frac{4(c-1)}{\pi^2 c^2}\left\{1 + \frac{8}{\pi^2}\left(\frac{1}{c}\right) + \frac{80}{\pi^4}\left(\frac{1}{c}\right)^2 + \left(\frac{2688-32\pi^2}{3\pi^6}\right)\left(\frac{1}{c}\right)^3 + \left(\frac{32256-896\pi^2}{3\pi^8}\right)\left(\frac{1}{c}\right)^4 + \dots\right\} \tag{87}$$

Another useful quantity is

$$K = D\frac{dk_o^2}{dc} = \frac{2(c-1)(1+k_o^2)}{c(1-c+k_o^2)} \tag{88}$$

Special values are:

$$\left.\begin{array}{lcl} c = 0 & : & K = 0 \\ c = 1 & : & K = 1 \\ c = \infty & : & K = 2 \end{array}\right\} \tag{89}$$

Expansions are given by:

c << 1

$$K = \frac{8}{c^2} e^{-\frac{2}{c}}(1-c)\left\{1 + \frac{8-2c}{c} e^{-\frac{2}{c}} + \frac{72-36c+3c^2}{c^2} e^{-\frac{4}{c}} + \frac{2048-1536c+288c^2-12c^3}{3c^3} e^{-\frac{6}{c}} + \dots\right\} \tag{90}$$

|c-1| << 1

$$K = 1 - \frac{4}{5}(1-c) - \frac{104}{175}(1-c)^2 - \frac{68}{175}(1-c)^3 \dots \tag{91}$$

c >> 1

$$K = 2\left\{1 - \left(1 - \frac{4}{\pi}\right)\frac{1}{c} - \frac{4}{\pi^2}\left(1 - \frac{12}{\pi^2}\right)\left(\frac{1}{c}\right)^2 \dots\right\}$$

The values of $\mathcal{K}_o$, k_o, k_o^2, $\frac{dk_o^2}{dc}$, D, $\frac{1}{\sqrt{3D}}$ and K are given in Table 8. Table 9 gives $\mathcal{K}_o$ and k_o at smaller intervals. K as a function of c for c < 1 is shown in Fig. 18.

c	K_o	$K_o^2 = -k_o^2$	$\frac{dk_o^2}{dc}$	D	$\frac{1}{\sqrt{3D}}$	K
0	1.000000	1.000000	.000000	1.000000	.577350	.000000
.1	1.000000	1.000000	0.164892(-5)	.900000	.608581	0.148403(-5)
.2	.999909	.999818	.009094	.800145	.645438	.007277
.3	.997414	.994834	.116201	.703635	.688281	.081763
.4	.985624	.971454	.373272	.617631	.734641	.230544
.5	.957504	.916814	.731896	.545367	.781799	.399152
.6	.907332	.823252	1.145954	.485878	.828277	.556794
.7	.828635	.686636	1.590035	.436913	.873458	.694706
.8	.710412	.504685	2.051119	.396287	.917138	.812832
.9	.525430	.276076	2.522370	.362219	.959300	.913650
.92	.474002	.224678	2.617473	.356065	.967553	.931990
.94	.413976	.171376	2.712805	.350108	.975751	.949774
.96	.340829	.116165	2.808348	.344339	.983889	.967024
.98	.242983	.059041	2.904080	.338750	.991974	.983759
.99	.172511	.029760	2.952020	.336021	.995993	.991940
1.00	.000000	.000000	3.000000	.333333	1.000000	1.000000

POINT SOURCE IN INFINITE MEDIUM

PART I

TABLE 8

1.0	.00000	.00000	3.00000	.33333	1.00000	1.00000
1.1	.56927	.32406	3.48181	.30858	1.03933	1.07443
1.2	.83454	.69646	3.96647	.28717	1.07739	1.13904
1.3	1.05708	1.11742	4.45313	.26848	1.11426	1.19555
1.4	1.25981	1.58713	4.94123	.25203	1.15005	1.24532
1.5	1.45110	2.10570	5.43039	.23745	1.18482	1.28945
1.6	1.63500	2.67323	5.92036	.22445	1.21866	1.32881
1.7	1.81378	3.28979	6.41094	.21278	1.25163	1.36411
1.8	1.98883	3.95544	6.90200	.20225	1.28378	1.39595
1.9	2.16107	4.67020	7.39343	.19271	1.31518	1.42480
2.0	2.33112	5.43413	7.88518	.18402	1.34587	1.45105
2.1	2.49945	6.24725	8.37715	.17608	1.37590	1.47503
2.2	2.66638	7.10957	8.86934	.16879	1.40530	1.49703
2.3	2.83216	8.02113	9.36173	.16207	1.43412	1.51728
2.4	2.99699	8.98192	9.85421	.15587	1.46238	1.53597
2.5	3.16101	9.99198	10.3469	.15012	1.49011	1.55328
2.6	3.32435	11.05130	10.8396	.14478	1.51735	1.56935
2.7	3.48710	12.15990	11.3324	.13980	1.54412	1.58431
2.8	3.64935	13.31778	11.8252	.13516	1.57043	1.59828
2.9	3.81116	14.52495	12.3182	.13081	1.59632	1.61134
3.0	3.97258	15.78142	12.8112	.12673	1.62180	1.62358
3.2	4.29445	18.44227	13.7973	.11929	1.67161	1.64590
3.4	4.61523	21.30036	14.7836	.11265	1.72000	1.66573
3.6	4.93515	24.35571	15.7700	.10675	1.76707	1.68346
3.8	5.25437	27.60835	16.7564	.10142	1.81293	1.69942
4.0	5.57300	31.05826	17.7429	.09659	1.85767	1.71384
4.5	6.36761	40.54642	20.2095	.08632	1.96509	1.74451
5.0	7.16016	51.26791	22.6763	.07802	2.06696	1.76925
5.5	7.95128	63.22285	25.1433	.07118	2.16406	1.78963
6.0	8.74136	76.41132	27.6106	.06544	2.25701	1.80670
6.5	9.53065	90.83335	30.0779	.06055	2.34628	1.82121
7.0	10.31935	106.48894	32.5448	.05634	2.43229	1.83370
7.5	11.10757	123.37818	35.0118	.05269	2.51537	1.84456
8.0	11.89542	141.50102	37.4790	.04947	2.59579	1.85409
8.5	12.68296	160.85753	39.9467	.04663	2.67380	1.86252
9.0	13.47025	181.44766	42.4138	.04409	2.74960	1.87003
9.5	14.25733	203.27149	44.8813	.04182	2.82337	1.87676
10.0	15.04423	226.32895	47.3488	.03977	2.89526	1.88282
∞	∞	∞	∞	.00000	∞	2.00000

POINT SOURCE IN INFINITE MEDIUM
(Part 2)

PART 1 $c < 1$

c	K_o	ΔK_o	c	K_o	ΔK_o
.00	1.000000		.50	.957504	
.01	1.000000	-.000000	.51	.953571	-.003933
.02	1.000000	-.000000	.52	.949413	-.004158
.03	1.000000	-.000000	.53	.945024	-.004389
.04	1.000000	-.000000	.54	.940398	-.004626
.05	1.000000	-.000000	.55	.935529	-.004869
.06	1.000000	-.000000	.56	.930412	-.005117
.07	1.000000	-.000000	.57	.925041	-.005371
.08	1.000000	-.000000	.58	.919408	-.005633
.09	1.000000	-.000000	.59	.913508	-.005900
.10	1.000000	-.000000	.60	.907332	-.006176
.11	1.000000	-.000000	.61	.900875	-.006457
.12	1.000000	-.000000	.62	.894128	-.006747
.13	1.000000	-.000000	.63	.887083	-.007045
.14	.999999	-.000001	.64	.879732	-.007351
.15	.999997	-.000002	.65	.872065	-.007667
.16	.999993	-.000004	.66	.864073	-.007992
.17	.999984	-.000009	.67	.855746	-.008327
.18	.999970	-.000014	.68	.847072	-.008674
.19	.999946	-.000024	.69	.838039	-.009033
.20	.999909	-.000037	.70	.828635	-.009404
.21	.999854	-.000055	.71	.818846	-.009789
.22	.999774	-.000080	.72	.808656	-.010190
.23	.999664	-.000110	.73	.798050	-.010606
.24	.999517	-.000147	.74	.787010	-.011040
.25	.999326	-.000191	.75	.775516	-.011494
.26	.999081	-.000245	.76	.763548	-.011968
.27	.998776	-.000305	.77	.751080	-.012468
.28	.998402	-.000374	.78	.738089	-.012991
.29	.997951	-.000451	.79	.724543	-.013546
.30	.997414	-.000537	.80	.710412	-.014131
.31	.996783	-.000631	.81	.695658	-.014754
.32	.996050	-.000733	.82	.680241	-.015417
.33	.995208	-.000842	.83	.664113	-.016128
.34	.994248	-.000960	.84	.647220	-.016893
.35	.993164	-.001084	.85	.629501	-.017719
.36	.991947	-.001217	.86	.610884	-.018617
.37	.990591	-.001356	.87	.591281	-.019603
.38	.989090	-.001501	.88	.570591	-.020690
.39	.987436	-.001654	.89	.548692	-.021899
.40	.985624	-.001812	.90	.525430	-.023262
.41	.983647	-.001977	.91	.500615	-.024815
.42	.981499	-.002148	.92	.474002	-.026613
.43	.979175	-.002324	.93	.445270	-.028732
.44	.976669	-.002506	.94	.413976	-.031294
.45	.973976	-.002693	.95	.379485	-.034491
.46	.971089	-.002887	.96	.340829	-.038656
.47	.968004	-.003085	.97	.296381	-.044448
.48	.964715	-.003289	.98	.242983	-.053398
.49	.961217	-.003498	.99	.172511	-.070472
.50	.957504	-.003713	1.00	.000000	-.172511

PART II c ≥ 1

c	k_o	Δk_o	c	k_o	Δk_o
1.00	.000000		1.50	1.451104	
1.01	.173897	.173897	1.51	1.469779	.018675
1.02	.246902	.073005	1.52	1.488385	.018606
1.03	.303582	.056680	1.53	1.506923	.018538
1.04	.351915	.048333	1.54	1.525397	.018474
1.05	.394979	.043064	1.55	1.543808	.018411
1.06	.434343	.039364	1.56	1.562159	.018351
1.07	.470937	.036594	1.57	1.580452	.018293
1.08	.505364	.034427	1.58	1.598689	.018237
1.09	.538039	.032675	1.59	1.616872	.018183
1.10	.569265	.031226	1.60	1.635003	.018131
1.11	.599268	.030003	1.61	1.653083	.018080
1.12	.628224	.028956	1.62	1.671114	.018031
1.13	.656272	.028048	1.63	1.689098	.017984
1.14	.683524	.027252	1.64	1.707036	.017938
1.15	.710071	.026547	1.65	1.724929	.017893
1.16	.735991	.025920	1.66	1.742780	.017851
1.17	.761346	.025355	1.67	1.760589	.017809
1.18	.786193	.024847	1.68	1.778358	.017769
1.19	.810577	.024384	1.69	1.796087	.017729
1.20	.834540	.023963	1.70	1.813779	.017692
1.21	.858117	.023577	1.71	1.831433	.017654
1.22	.881339	.023222	1.72	1.849052	.017619
1.23	.904233	.022894	1.73	1.866636	.017584
1.24	.926825	.022592	1.74	1.884186	.017550
1.25	.949135	.022310	1.75	1.901703	.017517
1.26	.971183	.022048	1.76	1.919189	.017486
1.27	.992986	.021803	1.77	1.936643	.017454
1.28	1.014561	.021575	1.78	1.954067	.017424
1.29	1.035922	.021361	1.79	1.971462	.017395
1.30	1.057082	.021160	1.80	1.988828	.017366
1.31	1.078053	.020971	1.81	2.006166	.017338
1.32	1.098845	.020792	1.82	2.023477	.017311
1.33	1.119469	.020624	1.83	2.040761	.017284
1.34	1.139934	.020465	1.84	2.058020	.017529
1.35	1.160248	.020314	1.85	2.075253	.017233
1.36	1.180419	.020171	1.86	2.092462	.017209
1.37	1.200455	.020036	1.87	2.109647	.017185
1.38	1.220362	.019907	1.88	2.126809	.017162
1.39	1.240146	.019784	1.89	2.143948	.017139
1.40	1.259813	.019667	1.90	2.161065	.017117
1.41	1.279370	.019557	1.91	2.178161	.017096
1.42	1.298819	.019449	1.92	2.195235	.017074
1.43	1.318168	.019349	1.93	2.212288	.017053
1.44	1.337419	.019251	1.94	2.229321	.017033
1.45	1.356577	.019158	1.95	2.246335	.017014
1.46	1.375646	.019069	1.96	2.263329	.016994
1.47	1.394630	.018984	1.97	2.280305	.016976
1.48	1.413532	.018902	1.98	2.297262	.016957
1.49	1.432356	.018824	1.99	2.314201	.016939
1.50	1.451104	.018748	2.00	2.331122	.016921

c	k_o	Δk_o	c	Δk_o	k_o
2.00	2.331122		2.50	3.161009	
2.01	2.348027	.016905	2.51	3.177372	.016363
2.02	2.364914	.016887	2.52	3.193728	.016356
2.03	2.381785	.016871	2.53	3.210078	.016350
2.04	2.398640	.016855	2.54	3.226421	.016343
2.05	2.415479	.016839	2.55	3.242758	.016337
2.06	2.432303	.016824	2.56	3.259088	.016330
2.07	2.449111	.016808	2.57	3.275413	.016325
2.08	2.465905	.016794	2.58	3.291731	.016318
2.09	2.482684	.016779	2.59	3.308043	.016312
2.10	2.499449	.016765	2.60	3.324349	.016306
2.11	2.516200	.016751	2.61	3.340650	.016301
2.12	2.532938	.016738	2.62	3.356944	.016294
2.13	2.549662	.016724	2.63	3.373233	.016289
2.14	2.566373	.016711	2.64	3.389517	.016284
2.15	2.583071	.016698	2.65	3.405795	.016278
2.16	2.599757	.016686	2.66	3.422067	.016272
2.17	2.616430	.016673	2.67	3.438334	.016267
2.18	2.633091	.016661	2.68	3.454596	.016262
2.19	2.649741	.016650	2.69	3.470853	.016257
2.20	2.666378	.016637	2.70	3.487104	.016251
2.21	2.683004	.016626	2.71	3.503351	.016247
2.22	2.699619	.016615	2.72	3.519592	.016241
2.23	2.716223	.016604	2.73	3.535829	.016237
2.24	2.732816	.016593	2.74	3.552060	.016231
2.25	2.749399	.016583	2.75	3.568287	.016227
2.26	2.765971	.016572	2.76	3.584510	.016223
2.27	2.782533	.016562	2.77	3.600727	.016217
2.28	2.799085	.016552	2.78	3.616940	.016213
2.29	2.815627	.016542	2.79	3.633149	.016209
2.30	2.832159	.016532	2.80	3.649353	.016204
2.31	2.848682	.016523	2.81	3.665553	.016200
2.32	2.865196	.016514	2.82	3.681748	.016195
2.33	2.881700	.016505	2.83	3.697939	.016191
2.34	2.898195	.016495	2.84	3.714126	.016187
2.35	2.914682	.016487	2.85	3.730309	.016183
2.36	2.931159	.016477	2.86	3.746487	.016178
2.27	2.947628	.016469	2.87	3.762662	.016175
2.38	2.964089	.016461	2.88	3.778832	.016170
2.39	2.980542	.016453	2.89	3.794999	.016167
2.40	2.996986	.016444	2.90	3.811161	.016162
2.41	3.013422	.016436	2.91	3.827320	.016159
2.42	3.029850	.016428	2.92	3.843475	.016155
2.43	3.046271	.016421	2.93	3.859626	.016151
2.44	3.062684	.016413	2.94	3.875774	.016148
2.45	3.079090	.016406	2.95	3.891917	.016143
2.46	3.095488	.016398	2.96	3.908058	.016141
2.27	3.111879	.016391	2.97	3.924194	.016136
2.48	3.128262	.016383	2.98	3.940327	.016133
2.49	3.144639	.016377	2.99	3.956457	.016130
2.50	3.161009	.016370	3.00	3.972583	.016126

c	k_o	Δk_o	c	k_o	Δk_o
3.00	3.972583		5.00	7.160161	
3.10	4.133665	.161082	5.10	7.318484	.158323
3.20	4.294446	.160781	5.20	7.476755	.158271
3.30	4.454958	.160512	5.30	7.634976	.158221
3.40	4.615231	.160273	5.40	7.793150	.158174
3.50	4.775288	.160057	5.50	7.951280	.158130
3.60	4.935150	.159862	5.60	8.109369	.158089
3.70	5.094837	.159687	5.70	8.267419	.158050
3.80	5.254365	.159528	5.80	8.425432	.158013
3.90	5.413747	.159382	5.90	8.583411	.157979
4.00	5.572996	.159249	6.00	8.741357	.157946
4.10	5.732124	.159128	6.10	9.530653	.789296
4.20	5.891141	.159017	6.20	10.319348	.788695
4.30	6.050055	.158914	6.30	11.107573	.788225
4.40	6.208875	.158820	6.40	11.895420	.787847
4.50	6.367607	.158732	6.50	12.682962	.787542
4.60	6.526258	.158651	6.60	13.470251	.787289
4.70	6.684834	.158576	6.70	14.257331	.787080
4.80	6.843340	.158506	6.80	15.044233	.786902
4.90	7.001781	.158441	6.90	15.830985	.786752
5.00	7.160161	.158380	7.00	16.617608	.786623

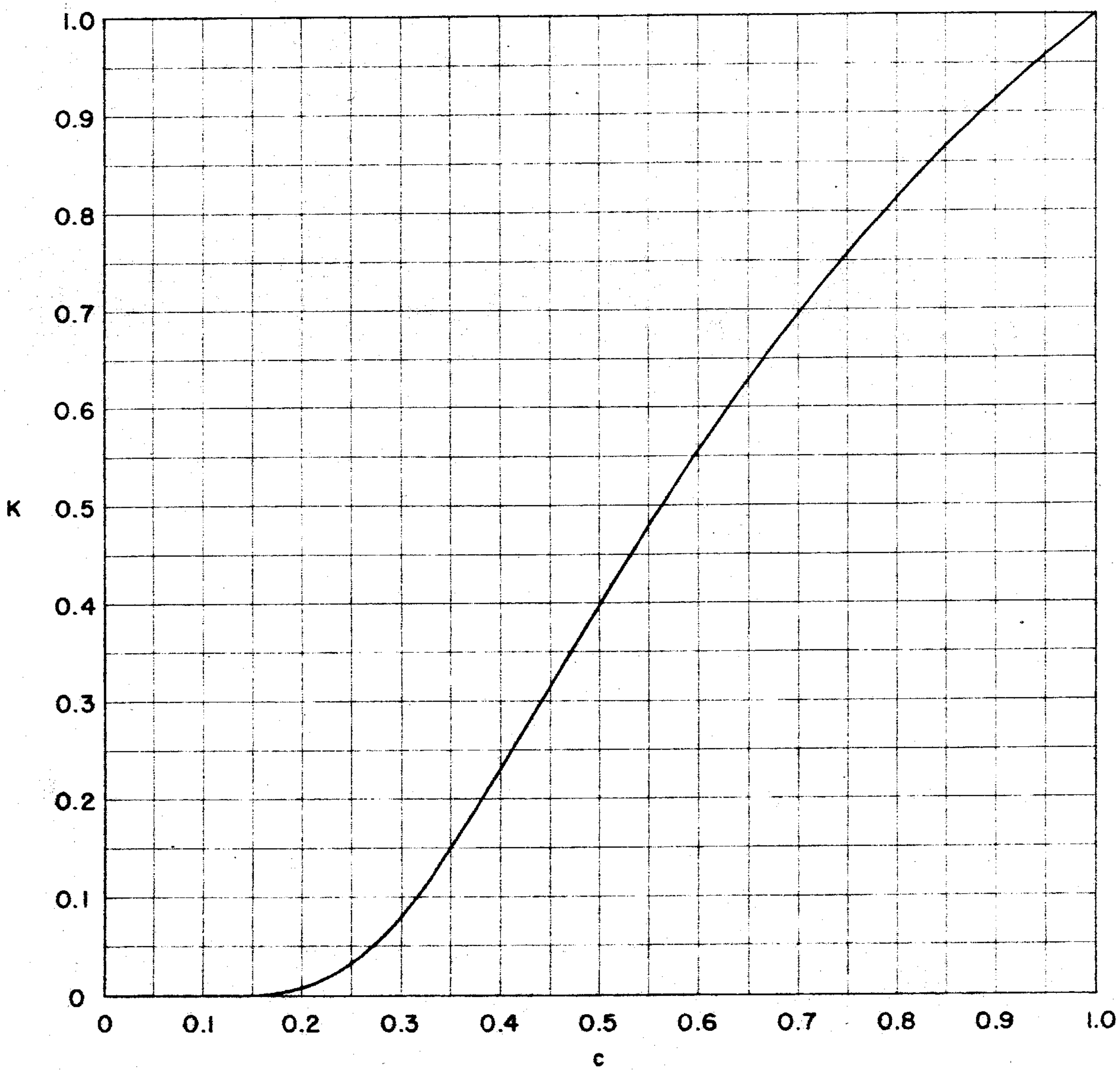


Fig. 18

14. Point Source Solution *

Under the assumption of isotropic scattering with constant c and σ the integral equation (12-4) for the neutron density in the presence of an isotropic source can be simplified to

$$\rho(\vec{r}) = \int \left\{ q_o(\vec{r}\,') + c\,\rho(\vec{r}\,') \right\} \frac{e^{-|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|^2} \, d\vec{r}\,' \qquad (1)$$

The solution, $\rho_p(\vec{r} - \vec{r}_o)$, corresponding to a unit isotropic point source at $\vec{r}_o$ (i.e., $q_o = \delta(\vec{r} - \vec{r}_o)$) is the Green's function of (1). Thus the density at $\vec{r}$, $\rho_{q_o}(\vec{r})$, due to an arbitrary distribution of isotropic sources $q_o(\vec{r}\,')$ is given by

$$\rho_{q_o}(\vec{r}) = \int \rho_p(\vec{r} - \vec{r}\,')\, q_o(\vec{r}\,')\, d\vec{r}\,' \qquad (2)$$

---

* Results concerning the point and plane source solution have been derived by several authors in a more or less similar way.

(1) W. Bothe, Einige Diffusions Probleme, Zeit. f. Phys. 118, 401, (1941).

(2) G. Placzek, Unpublished Report to the Briggs Committee, 1941.

(3) G. Placzek, Diffusion of Thermal Neutrons, Phys. Rev. 60, A166, (1941).

(4) E. P. Wigner, The Diffusion of Slow Neutrons in Absorbing Materials, Classified Report A20, issued August 11, 1941.

(5) W. Bothe, Die Diffusion von einer Punktquelle Aus, Zeit. f. Phys. 119, 493, (1942).

(6) B. Davison and R. Peierls, Density Distribution near a Point Source, British Declassified Report MS76, 1943.

(7) G. Placzek and G. M. Volkoff, Notes on Diffusion of Neutrons Without Change in Energy, Canadian Declassified Report MT4, (NRC-1548), 1943.

(8) E. P. Wigner, Solution of Boltzmann's Equation for Monoenergetic Neutrons in an Infinite Homogeneous Medium, Classified Report CP1120 issued November 30, 1943.

The contents of this and the subsequent section go considerably beyond what may be found in these papers. The numerical results presented here are an extension of work by Bengt Carlson contained in unpublished Montreal computing reports, in which also some of the analytical expressions derived below have been given for the first time, as well as of AECD-1943 (LADC-506) entitled, "Neutron Density, Point Source and Plane Source" by M. Goldstein et al.

where ρ_p satisfies the equation

$$\rho(\vec{r}) = \int \left\{ \delta(\vec{r}\,') + c\rho(\vec{r}\,') \right\} \frac{e^{-|\vec{r} - \vec{r}\,'|}}{4\pi |\vec{r} - \vec{r}\,'|^2} \, d\vec{r}\,' \tag{3}$$

Here the subscript "p" has been dropped for convenience.

In an infinite medium with $c > 1$ there is, of course, no steady state solution when sources are present. A neutron coming from the source produces more than one neutron which in turn each produces more than one neutron. Consequently the neutron density is everywhere continuously increasing. However we will still carry through the analysis in this case. In section 11 it was shown that the "steady state" solution is the Laplace transform of the solution of the time dependent problem. The latter will have a meaning also for $c > 1$. Moreover, the infinite medium Green's function is of importance in the treatment of problems involving finite media. For these too, there can exist stationary solutions though c be larger than one.

For $c < 1$ the quadratically integrable solution of (3) may be obtained by taking Fourier transforms. The general solution of (3) is then found by adding the general solution (13-7) of the homogeneous equation. For $c > 1$ the Fourier transform of ρ will be shown to have poles on the real axis in the space of the transform variable. As shown in Appendix C a particular solution of (3) is obtained by specifying a path around the pole. The general solution may then be found as above.

If $\phi(\vec{k})$ denotes the Fourier transform of $\rho(\vec{r})$ we have

$$\phi(\vec{k}) = \int e^{i\vec{k}\cdot\vec{r}} \rho(\vec{r}) \, d\vec{r} \tag{4}$$

Taking the Fourier transform of (3) yields:

$$\phi(\vec{k}) = \left\{ c\,\phi(\vec{k}) + 1 \right\} \frac{\tan^{-1} k}{k} \tag{5}$$

or $$\phi(\vec{k}) = \frac{\tan^{-1}k/k}{1-c\,\frac{\tan^{-1}k}{k}} \tag{6a}$$

$$= \frac{1}{\frac{k}{\tan^{-1}k} - c} \tag{6b}$$

Expanding (6a) in powers of c gives:

$$\phi(\vec{k}) = \sum_{n=0}^{\infty} c^n \left(\frac{\tan^{-1}k}{k}\right)^{n+1} \tag{7}$$

Remembering the Convolution theorem and that $\frac{\tan^{-1} k}{k}$ is the Fourier transform of $e^{-r}/4\pi r^2$ it is apparent that the n'th term in (7) is just the Fourier transform of the portion of the neutron density due to neutrons having suffered n collisions after leaving the source. Alternatively said the n'th term in (7) is the transform of the n'th term of the Neumann solution (11-8).

With (6) we can now determine the even moments of $\rho(\vec{r})$. Since ϕ depends only on the absolute value of $\vec{k}$, $\rho(\vec{r})$ will be spherically symmetric (i.e., $\rho = \rho(r)$). Hence the relation between ϕ and ρ simplifies to:

$$\phi(k) = \int \rho(r) \frac{\sin kr}{kr} \, d\vec{r} \tag{8}$$

We know that any function may be expanded in a power series valid till the first singularity. In Appendix C it is shown that the nearest singularity to the origin of $\phi(k)$ is the pole corresponding to the vanishing of the denominator in (6a). For $c < 1$ this pole is on the imaginary axis at a finite distance from the origin. When $c = 1$, the pole is at the origin. It lies on the axis of reals for $c > 1$. Thus, if $c \neq 1$, one obtains on expanding in powers of k:

$$\phi(k) = \sum_{n=0}^{\infty} \frac{k^{2n}(-1)^n}{(2n+1)\,!} \int r^{2n} \rho(r) \, d\vec{r} \tag{9}$$

(9) gives the even moments of ρ in terms of the coefficients in the expansion of $\phi(k)$. In this form the failure of the above considerations at $c = 1$ is physically understandable. For $c < 1$ absorption will result in a decrease in

the number of neutrons after a collision. The neutron density will then decrease exponentially at large distances and the moments will exist. When $c = 1$ there is no absorption and the neutron density will extend to infinity. Infinite moments would then be expected. If $c > 1$ the moments of ρ exist formally in the Abelian sense since the distribution then oscillates in sign at large distances. Some of these moments are negative and hence rather unphysical. We will use them only for mathematical purposes.

Expanding (6b) gives:

$$\phi(k) = \frac{1}{1-c+\frac{k^2}{3}+\ldots} = \frac{1}{1-c} - \frac{1}{3(1-c)^2} k^2 + \ldots \qquad (10)$$

which may be written as

$$\phi(k) = \sum_{m=0}^{\infty} \beta_m (-k^2)^m \qquad (11)$$

where

$$\beta_o = \frac{1}{1-c} \qquad (12)$$

$$\beta_m = \frac{1}{(1-c)^2} \sum_{s=1}^{m} b_{m,s} \left(\frac{c}{1-c}\right)^{s-1} \qquad (13)$$

and

$$b_{m,1} = \frac{1}{2m+1} \qquad (14)$$

$$b_{m,s+1} = \sum_{n=s}^{m-1} \frac{b_{n,s}}{1+2(m-n)} \qquad (15)$$

Table 10 gives the coefficients $b_{m,s}$ for $m=1$ to $m=10$. Comparing (11) and (9) we see that the 2n'th moment of the density which is defined as

$$M_{2n} = \int r^{2n} \rho(r)\, d\vec{r} \qquad (16)$$

is given by:

$$M_{2n} = \beta_n (2n+1)! \qquad (17)$$

As examples we note that from (10) it follows

$$\int \rho\, d\vec{r} = \frac{1}{1-c} \qquad (18)$$

$$\int r^2 \rho\, d\vec{r} = \frac{2}{(1-c)^2} \qquad (19)$$

s \ m	1	2	3	4	5	6	7	8	9	10
1	.333 3333	.200 0000	.142 8571	.111 1111	.090 909 09	.076 923 08	.066 666 67	.058 823 53	.052 631 58	.047 619 05
2		.111 1111	.133 3333	.135 2381	.131 216 9	.125 458 7	.119 391 7	.113 533 3	.108 062 4	.103 023 2
3			.037 0370 4	.066 6666 7	.087 619 05	.102 179 9	.112 298 5	.119 328 9	.124 183 3	.127 479 9
4				.012 3456 8	.029 629 63	.047 830 69	.065 222 81	.081 160 23	.095 478 19	.108 218 9
5					.004 115 226	.012 345 68	.023 633 16	.036 911 62	.051 345 44	.066 333 45
6						.001 371 742	.004 938 272	.010 934 74	.019 251 42	.029 619 49
7							.000 457 247 4	.001 920 439	.004 828 532	.009 461 972
8								.000 152 415 8	.000 731 595 8	.002 058 920
9									.000 050 805 26	.000 274 348 4
10										.000 016 935 09

$$b_{m,1} = \frac{1}{2m+1}$$

$$b_{m,s+1} = \sum_{n=s}^{m-1} \frac{b_{ns}}{1+2(m-n)}$$

VALUES OF $b_{m,s}$

POINT SOURCE IN INFINITE MEDIUM

TABLE 10

and thus that

$$(r^2)_{av} = \frac{\int r^2 \rho \, d\vec{r}}{\int \rho \, d\vec{r}} = \frac{2}{1 - c} \tag{20}$$

(20) expresses the obvious fact that the larger c (i.e., the less absorption there is), the farther a neutron will go on the average.

Since the density in (18) is normalized to a source emitting one neutron per second it yields, on normalizing to Q neutrons per second and returning to ordinary units, the perspicuous result

$$\int \rho \, d\vec{r} = \frac{Q}{\sigma v(1 - c)} = \frac{Q}{\sigma_a v} \tag{21}$$

where $\sigma_a = \sigma(1 - c)$ is the absorption cross section. Thus (21) states the physically expected result that the total number of neutrons present is the number emitted per second by the source times the mean lifetime for absorption.

To obtain $\rho(r)$ we use the inversion formula

$$\rho(r) = \frac{1}{(2\pi)^3} \int e^{-i\vec{k}\cdot\vec{r}} \phi(k) \, d\vec{k}$$

$$= \frac{1}{(2\pi^2)r} \int_0^\infty \phi(k) \, k \sin kr \, dk \tag{22}$$

Before evaluating this integral let us examine the behavior of ρ for small r. This can be done in a manner similar to the above where the even moments of ρ were related to the behavior of ϕ in the neighborhood of the origin. There exist many Abelian and Tauberian theorems relating the asymptotic behavior of a function and its transform. We will prove a special case of one which will give us the dominant term in ρ for small r and then state the result of using the other theorems.

Multiplying (22) by r^2 and taking the limit $r \to 0$ gives:

$$\lim_{r \to 0} r^2 \rho(r) = \frac{1}{2\pi^2} \lim_{r \to 0} \int_0^\infty \phi(k)\, kr \sin kr \, dk \tag{23}$$

$$= \frac{1}{2\pi^2} \lim_{r \to 0} \int_0^\infty \phi\left(\frac{z}{r}\right) \frac{z \sin z \, dz}{r}$$

Since for large values of k

$$\phi(k) \sim \frac{\pi}{2k} \tag{24}$$

(23) gives:

$$\lim_{r \to 0} r^2 \rho(r) = \frac{1}{2\pi^2} \int_0^\infty \frac{\pi r}{2z} \frac{z \sin z \, dz}{r} \tag{25}$$

$$= \frac{1}{4\pi} \int_0^\infty \sin z \, dz = \frac{1}{4\pi}$$

Thus for r small enough

$$\rho \sim \frac{1}{4\pi r^2} \tag{26}$$

This is just what we would expect since the neutron density due to neutrons coming from the source without suffering a collision is

$$\rho = \frac{e^{-r}}{4\pi r^2} \tag{27}$$

Subtracting $\frac{1}{4\pi r^2}$ from ρ and applying similar theorems to the remainder gives the next most important terms in the expansion of ρ in the vicinity of the origin. Proceeding in this manner yields:

$$\rho(r) = \frac{1}{4\pi r^2} \left\{ 1 + \left(\frac{c\pi^2}{4} - 1\right) r + \left(2c - c^2 \frac{\pi^2}{4}\right) r^2 \log r \right\} + 0\,(\text{const}) \qquad r \ll 1 \tag{28}$$

Returning to the complete expression for the density we note that (22) can be written as

$$\rho(r) = \frac{1}{2\pi r} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\tan^{-1} k \; e^{ikr}}{1 - c \left(\frac{\tan^{-1} k}{k}\right)} \, dk \tag{29}$$

In Appendix C it is shown that this integral can be transformed giving

$$\rho(r) = \frac{1}{4\pi r}\left\{\frac{-\partial K_o^2}{\partial c}\, e^{-K_o r} + \int_o^1 g(c,\mu)\, e^{-r/\mu}\, \frac{d\mu}{\mu^2}\right\} \tag{30}$$

where K_o is the positive root of (13-68) and

$$g(c,\mu) = \frac{1}{(1 - c\mu \tanh^{-1}\mu)^2 + (\frac{\pi}{2} c\,\mu)^2} \tag{31}$$

Numerical values of $g(c,\mu)$ are given in Table 11. This function is graphed in Figure 19.

The general solution of (3) obtained by adding the general solution (13-7) of the homogeneous equation is

$$\rho(\vec{r}) = \frac{1}{4\pi r}\left\{\frac{-\partial K_o^2}{\partial c}\, e^{-K_o r} + \int_o^1 g(c,\mu)\, e^{-r/\mu}\, \frac{d\mu}{\mu^2}\right\} + \int f(\vec{u})\, e^{-K_o \vec{u}\cdot\vec{r}}\, du \tag{32}$$

with $f(\vec{u})$ an arbitrary function.

The general spherically symmetric solution is obtained from this by averaging over the direction of $\vec{r}$.

Since

$$\frac{1}{4\pi}\iint f(\vec{u})\, e^{-K_o \vec{u}\cdot\vec{r}}\, du\, d\Omega_R = \frac{\sinh K_o r}{K_o r}\int f(\vec{u})\, du = (\text{const})\,\frac{\sinh K_o r}{r} \tag{33}$$

and

$$e^{-K_o r} = \cosh K_o r - \sinh K_o r \tag{34}$$

We see that the general spherically symmetric point source solution is

$$\rho(r) = \frac{1}{4\pi r}\left\{\frac{-\partial K_o^2}{\partial c}\cosh K_o r + \lambda \sinh K_o r + \int_o^1 g(c,\mu)\, e^{-r/\mu}\, \frac{d\mu}{\mu^2}\right. \tag{35}$$

where λ is an arbitrary constant.

TABLE 11 (Sheet 1)

Values of g(c,μ)

$$g(c,\mu) = \frac{1}{(1-c\mu\tanh^{-1}\mu)^2 + (\frac{\pi}{2}c\mu)^2}$$

μ \ c	c=0	c=0.1	c=0.2	c=0.3	c=0.4	c=0.5	c=0.6	c=0.7	c=0.8	c=0.9	c=1.0
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.05	1.000000	1.000439	1.000754	1.000946	1.001015	1.000959	1.000780	1.000478	1.000051	.998831	.998831
.10	1.000000	1.001762	1.003032	1.003805	1.004079	1.003855	1.003131	1.001911	1.000198	.997997	.995314
.15	1.000000	1.003990	1.006874	1.008634	1.009257	1.009839	1.007084	1.004303	1.000414	.995444	.989425
.20	1.000000	1.007157	1.012356	1.015535	1.016656	1.015704	1.012692	1.007657	1.000657	.991776	.981116
.25	1.000000	1.011314	1.019586	1.024659	1.026437	1.024885	1.020033	1.011974	1.000860	.986895	.970324
.30	1.000000	1.016533	1.028716	1.036217	1.038826	1.036467	1.029208	1.017254	1.000930	.980664	.956959
.35	1.000000	1.022908	1.039952	1.050495	1.054125	1.050697	1.040349	1.023484	1.000735	.972905	.940906
.40	1.000000	1.030563	1.053567	1.067875	1.072740	1.067902	1.053620	1.030639	1.000095	.963383	.922015
.45	1.000000	1.039664	1.069927	1.088873	1.095214	1.088506	1.069219	1.038661	.998763	.951798	.900096
.50	1.000000	1.050429	1.089526	1.114189	1.122280	1.113068	1.087383	1.047444	.996396	.937756	.874907
.55	1.000000	1.063153	1.113042	1.144794	1.154945	1.142327	1.108388	1.056795	.992511	.920748	.846135
.60	1.000000	1.078243	1.141430	1.182065	1.194621	1.177276	1.132534	1.066373	.986423	.900094	.813371
.65	1.000000	1.096280	1.176083	1.228029	1.243357	1.219267	1.160114	1.075579	.977127	.874880	.776067
.70	1.000000	1.118125	1.219125	1.285812	1.304241	1.270177	1.191317	1.083337	.963119	.843836	.733466
.75	1.000000	1.145133	1.274000	1.360533	1.382188	1.332652	1.262782	1.087675	.942055	.805131	.684478
.80	1.000000	1.179610	1.346788	1.461329	1.485572	1.410425	1.297165	1.084798	.910092	.755983	.627430
.85	1.000000	1.225969	1.449657	1.606660	1.630111	1.508395	1.313164	1.066868	.860440	.691798	.559508
.90	1.000000	1.294427	1.612462	1.842025	1.849536	1.629844	1.313164	1.015674	.779605	.603895	.475236
.95	1.000000	1.419421	1.945044	2.331852	2.228849	1.743429	1.244393	.878172	.633330	.470747	.360398
1.00	1.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000

Values of g(c,μ)

$$g(c,\mu) \quad \frac{1}{(1-c\mu\tanh^{-1}u)^2+(\frac{\pi}{2}c\mu)^2}$$

μ \ c	c=1.1	c=1.2	c=1.3	c=1.4	c=1.5	c=1.6	c=1.7	c=1.8	c=1.9	c=2.0	c=2.1
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.05	.998037	.997122	.996085	.994929	.993654	.992260	.990748	.989121	.987378	.985522	.983553
.10	.992158	.988538	.984463	.979946	.974999	.969635	.963870	.957717	.951195	.944319	.937106
.15	.982395	.974399	.965487	.955712	.945130	.933802	.921789	.909155	.895961	.882272	.868149
.20	.968796	.954947	.939714	.923245	.905694	.887216	.867962	.848080	.827711	.806989	.786037
.25	.951426	.930501	.907860	.883815	.858669	.832715	.806220	.779431	.752568	.725823	.699362
.30	.930362	.901439	.870748	.838819	.806139	.773142	.740204	.707639	.675707	.644608	.614499
.35	.905687	.868180	.829251	.789669	.750091	.711055	.672982	.636190	.600902	.567265	.535362
.40	.877482	.831160	.784239	.737695	.692282	.648554	.606887	.567509	.530535	.495991	.463842
.45	.845819	.790811	.736536	.684072	.634153	.587226	.543513	.503072	.465845	.431699	.400453
.50	.810749	.747539	.686878	.629783	.576807	.528162	.483821	.443606	.407255	.374458	.344895
.55	.772287	.701701	.635889	.575594	.521020	.472025	.428267	.389303	.354656	.323854	.296451
.60	.730390	.653580	.584056	.522052	.467269	.419129	.376942	.340000	.307630	.279224	.254241
.65	.684926	.603354	.531710	.469490	.415774	.369514	.329675	.295311	.265594	.239810	.217357
.70	.635621	.551057	.479002	.418024	.366530	.323005	.286114	.254721	.227886	.204834	.184932
.75	.581968	.496511	.425856	.367542	.319310	.279242	.245766	.217622	.193809	.173529	.156149
.80	.523051	.439185	.371877	.317640	.273643	.237665	.207993	.183314	.162617	.145122	.130223
.85	.457140	.377900	.316122	.267456	.228677	.197414	.171926	.150924	.133445	.118764	.106328
.90	.380575	.309974	.256435	.215134	.182749	.156966	.136153	.119138	.105067	.093311	.083395
.95	.283266	.227745	.186685	.155583	.131520	.112556	.097366	.085021	.074861	.066403	.059290
1.00	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000

Values of g(c,μ)

$$g(c,\mu) = \frac{1}{(1-c\mu\tanh^{-1}\mu)^2\ (\frac{\pi}{2}c\mu)^2}$$

μ \ c	c=2.2	c=2.3	c=2.4	c=2.5	c=2.6	c=2.7	c=2.8	c=2.9	c=3.0
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.05	.981473	.979283	.976986	.974581	.972072	.969459	.966745	.963932	.961021
.10	.929574	.921741	.913625	.905244	.896618	.887763	.878699	.869444	.860015
.15	.853654	.838846	.823781	.808515	.793099	.777582	.762009	.746422	.730862
.20	.764970	.743890	.722890	.702054	.681451	.661145	.641188	.621624	.602489
.25	.673324	.647824	.622954	.598786	.575373	.552753	.530951	.509978	.489839
.30	.585492	.557661	.531052	.505683	.481556	.458653	.436948	.416402	.396974
.35	.505224	.476845	.450190	.425202	.401810	.379936	.359495	.340401	.322569
.40	.434005	.406371	.380813	.357196	.335382	.315235	.296624	.279425	.263522
.45	.371906	.345844	.322056	.300339	.280503	.262368	.245774	.230570	.216623
.50	.318250	.294223	.272537	.252940	.235203	.219124	.204522	.191237	.179127
.55	.272042	.250262	.230789	.213340	.197668	.183559	.170824	.159304	.148856
.60	.232211	.212729	.195450	.180075	.166354	.154070	.143040	.133107	.124135
.65	.197730	.180505	.165329	.151906	.139989	.129371	.119876	.111357	.103689
.70	.167633	.152605	.139414	.127804	.117541	.108432	.100315	.093054	.086536
.75	.141164	.128169	.116838	.106907	.098161	.090422	.083546	.077410	.071915
.80	.117446	.106417	.096839	.088472	.081126	.074643	.068896	.063779	.059203
.85	.095711	.086582	.078679	.071795	.065765	.060456	.055757	.051580	.047851
.90	.974960	.067729	.061484	.056057	.051312	.047140	.043454	.040180	.037261
.95	.053254	.048089	.043637	.039772	.036396	.033431	.030812	.028488	.026417
1.00	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000	.000000

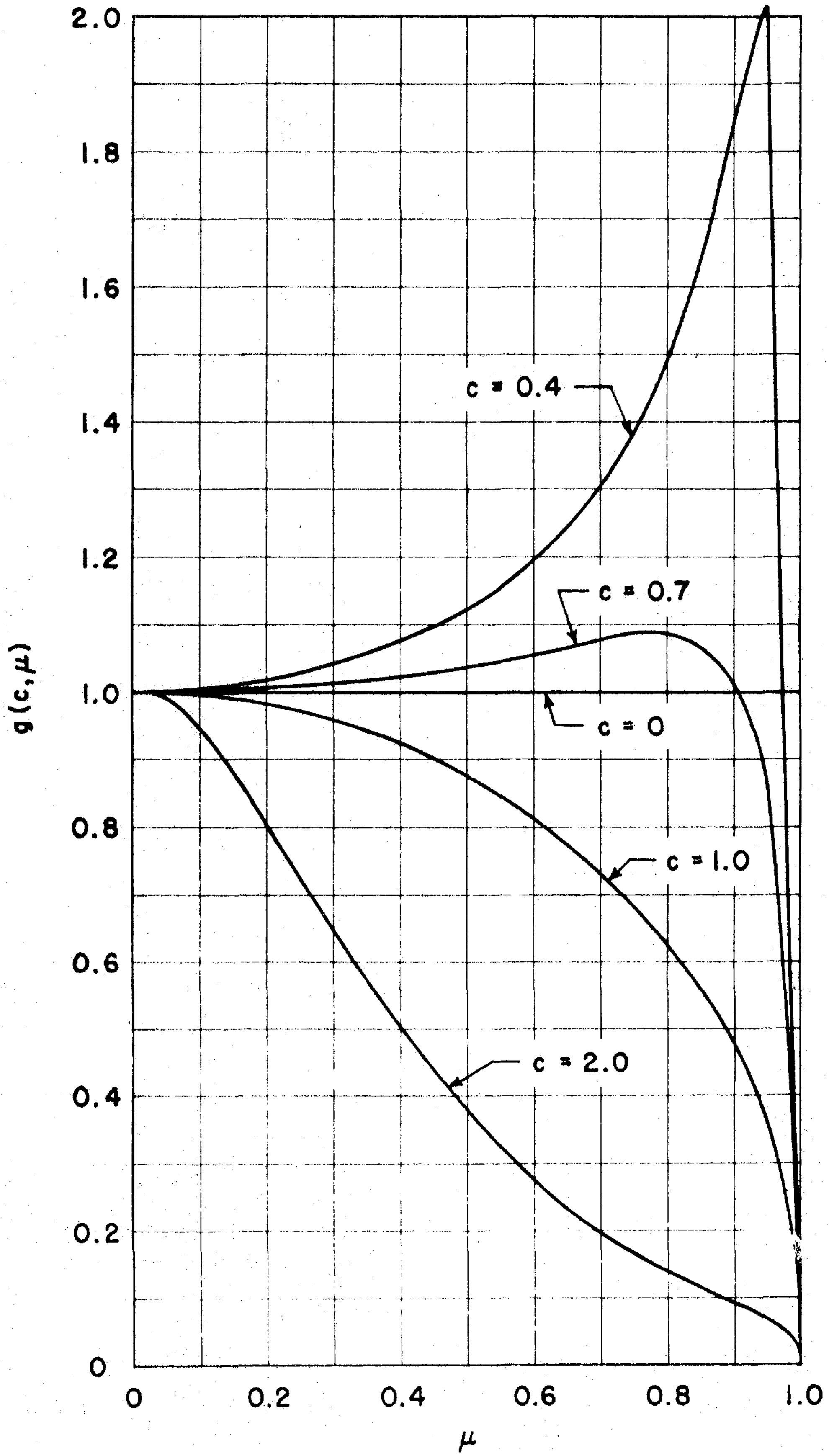


Fig. 19
THE FUNCTION $g(c, \mu)$

It should be noted that instead of using the averaging process of (33), the general spherically symmetric solution of (3) could also have been obtained directly by adding the general spherically symmetric infinite medium solution (13-35) to (30).

Returning to the particular point source solution (30), we note that for sufficiently large distances the first term will predominate since it vanishes less strongly than e^{-r} while all contributions from the integral vanish as e^{-r} or faster. Below it will be seen that for small distances this second term dominates and behaves almost like $e^{-r}/4\pi r^2$. For discussion it is then convenient to decompose ρ into

$$\rho(r) = p(r) + \rho_{as}(r) \tag{36}$$

where ρ_{as}, the asymptotic density, is the first part of (30), i.e.,

$$\rho_{as}(r) = \frac{-\partial K_o^2}{\partial c} \frac{e^{-K_o r}}{4\pi r} \tag{37}$$

and consequently

$$p(r) = \frac{1}{4\pi r} \int_0^1 g(c,\mu)\, e^{-r/\mu} \frac{d\mu}{\mu^2} \tag{38}$$

Defining $\epsilon(r)$ by

$$p(r) = \frac{e^{-r}}{4\pi r^2} \epsilon(r) \tag{39}$$

gives

$$\epsilon(r) = re^r \int_0^1 \frac{g(c,\mu)\, e^{-r/\mu}}{\mu^2} d\mu \tag{40}$$

Having previously obtained the even moments of the total density, we will now obtain those for the asymptotic and non-asymptotic densities. The even asymptotic density moments, A_{2n}, are defined by

$$A_{2n} = \int \rho_{as} r^{2n}\, d\vec{r} \tag{41}$$

Using (37) we obtain

$$A_{2n} = \frac{-\partial K_o^2}{\partial c} \int r^{2n} \frac{e^{-K_o r}}{4\pi r} d\vec{r} = \frac{(2n+1)!}{K_o^{2n+2}} \frac{\partial K_o^2}{\partial c} \tag{42}$$

In particular

$$\int \rho_{as}\, d\vec{r} = -\frac{1}{K_o^2} \frac{\partial K_o^2}{\partial c} \tag{43}$$

and

$$\int \rho_{as} r^2\, d\vec{r} = -\frac{6}{K_o^4} \frac{\partial K_o^2}{\partial c} \tag{44}$$

Hence

$$(r^2)_{av}\,(as) = \frac{\int \rho_{as} r^2\, d\vec{r}}{\int \rho_{as}\, d\vec{r}} = 6/K_o^2 \tag{45}$$

Using (18) we find:

$$\frac{\int \rho_{as}\, d\vec{r}}{\int \rho\, d\vec{r}} = -\frac{\partial K_o^2}{\partial c} \frac{(1-c)}{K_o^2} = K \tag{46}$$

Thus the quantity K gives the fraction of neutrons in the asymptotic distribution. Table 8 gives K as a function of c. This is plotted in Figure 18. Expansions for K are given by equations (13-90 to 13-92).

From Figure 18 we see that for pure scattering ($c = 1$) $K = 1$, and hence that all neutrons are in the asymptotic distribution. For c small (large absorption) K (c) is very flat and approximately zero. Practically all neutrons are then in the non-asymptotic distribution. Even for equal amounts of absorption and scattering ($c = 0.5$) K is only 0.4. Thus when there is any appreciable amount of absorption one must be very careful in trying to draw conclusions from the asymptotic distribution alone since this may describe but few of the neutrons. This is particularly true since most of the neutrons in the asymptotic distribution may be in the region where the non-asymptotic distribution is still large.

The even moments of the non-asymptotic density, N_{2n}, defined by

$$N_{2n} = \int p r^{2n}\, d\vec{r} \tag{47}$$

may be obtained by subtraction of the moments previously found.

Thus:

$$N_{2n} = \int (\rho - \rho_{as})\, r^{2n}\, d\vec{r} = M_{2n} - A_{2n} \qquad (48)$$

Incidentally this also determinesthe even moments with respect to μ of $g\,(c,\mu)$. Using (38)

$$\int p\, r^{2n}\, d\vec{r} = \int_0^1 \frac{g\,(c,\mu)\, d\mu}{\mu^2} \int_0^\infty r^{2n+1}\, e^{-r/\mu}\, dr \qquad (49)$$

$$= (2n+1)! \int_0^1 \mu^{2n}\, g\,(c,\mu)\, d\mu$$

or

$$T_{2n} = \int_0^1 \mu^{2n}\, g\,(c,\mu)\, d\mu = \frac{N_{2n}}{(2n+1)!} \qquad (50)$$

From (17) and (42) we find:

$$N_o = \frac{1 - K}{1 - c} \qquad (51)$$

and

$$N_2 = \frac{2}{(1 - c)^2} \quad (1 - 3\,K\,D) \qquad (52)$$

Thus:

$$(r^2)_{av}(\text{non-as}) = \frac{N_2}{N_b} = \frac{2}{1 - c}\;\frac{(1 - 3\,KD)}{1 - K} \qquad (53)$$

Table 12 gives $N_m/m!$ for even m between m = 0 and m = 20. N_o, N_2 and $(r^2)_{av}$(non-as) are plotted in Figures 20 and 21. The various averages of r^2 are given in Table 13 and plotted in Figure 22.

On looking at Figure 22 and Table 13 we see that for c small $(r^2)_{av}$ coincides very closely with $r^2{}_{av}$(non-as) in agreement with the previous observation that for large absorption most of the neutrons are in the non-asymptotic distribution. It should be noted also that $(r^2)_{av}$(non-as) ~ 2 (mean free path)^2 for $c < 1$. (For $c > 1$ it decreases slightly.) We see that it can be quite incorrect to consider the non-asymptotic distribution to be negligible after a mean free path from the source.

For a complete determination of the non-asymptotic distribution we must consider the function $g\,(c,\mu)$ of equation (31) (Table 11, Figure 19) in greater detail. This function is of great importance since it also occurs

TABLE 12
POINT SOURCE IN INFINITE MEDIUM

TABLE OF $N_m/m! = (m+1)\, T_m(c)$

N_m = m'th moment of non-asymptotic distribution

$T_m = \int_0^1 \mu^m\, g(c, \mu)\, d\mu$

PART 1

m \ c	0	0.1	0.2	0.3	0.4	0.5	0.6
0	1.0000	1.1111	1.2409	1.3118	1.2824	1.2017	1.1080
2	1.0000	1.2346	1.5352	1.6886	1.5912	1.3878	1.1775
4	1.0000	1.3108	1.7340	1.9366	1.7708	1.4735	1.1891
6	1.0000	1.3665	1.8887	2.1239	1.8924	1.5190	1.1822
8	1.0000	1.4108	2.0173	2.2749	1.9810	1.5438	1.1681
10	1.0000	1.4478	2.1285	2.4015	2.0485	1.5568	1.1512
12	1.0000	1.4796	2.2271	2.5103	2.1013	1.5624	1.1331
14	1.0000	1.5076	2.3161	2.6057	2.1434	1.5632	1.1150
16	1.0000	1.5327	2.3975	2.6904	2.1775	1.5607	1.0972
18	1.0000	1.5554	2.4727	2.7665	2.2053	1.5559	1.0799
20	1.0000	1.5762	2.5427	2.8354	2.2282	1.5495	1.0633

m \ c	0.7	0.8	0.9	1.0	1.1	1.2	1.3
0	1.0176	.9358	.8635	.8000	.7443	.6952	.6518
2	.9936	.8414	.7176	.6171	.5352	.4678	.4119
4	.9562	.7745	.6343	.5257	.4409	.3739	.3203
6	.9198	.7236	.5778	.4685	.3855	.[illegible]216	.2715
8	.8867	.6832	.5361	.4285	.3483	.2876	.2408
10	.8570	.6500	.5036	.3984	.3212	.2634	.2193
12	.8304	.6220	.4773	.3747	.3003	.2451	.2033
14	.8065	.5981	.4554	.3554	.2836	2307	.1908
16	.7848	.5772	.4368	.3393	.2698	.2189	.1807
18	.7651	.5590	.4204	.3256	.2582	.2090	.1723
20	.7471	.5428	.4066	.3138	.2483	.2007	.1652

TABLE 12

PART 2

m \ c	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	.6133	.5789	.5480	.5202	.4949	.4720	.4510
2	.3652	.3258	.2924	.2637	.2391	.2176	.1989
4	.2769	.2414	.2121	.1876	.1670	.1495	.1346
6	.2318	.1999	.1739	.1525	.1347	.1197	.1071
8	.2041	.1748	.1513	.1321	.1162	.1030	.0918
10	.1850	.1579	.1362	.1186	.1041	.0921	.0820
12	.1710	.1456	.1253	.1089	.0955	.0844	.0750
14	.1601	.1361	.1170	.1016	.0890	.0785	.0698
16	.1514	.1286	.1104	.0958	.0838	.0740	.0657
18	.1442	.1223	.1050	.0910	.0797	.0702	.0624
20	.1382	.1171	.1005	.0871	.0762	.0671	.0596

m \ c	2.1	2.2	2.3	2.4	2.5	2.6	2.7
0	.4318	.4142	.3979	.3828	.3688	.3558	.3437
2	.1825	.1680	.1552	.1438	.1335	.1244	.1161
4	.1217	.1105	.1008	.0923	.0848	.0781	.0722
6	.0963	.0870	.0790	.0720	.0659	.0605	.0558
8	.0823	.0742	.0673	.0612	.0559	.0513	.0472
10	.0734	.0661	.0598	.0544	.0497	.0455	.0418
12	.0671	.0604	.0546	.0497	.0453	.0415	.0382
14	.0624	.0562	.0508	.0461	.0421	.0385	.0354
16	.0588	.0528	.0478	.0434	.0396	.0362	.0333
18	.0558	.0501	.0453	.0412	.0375	.0344	.0316
20	.0533	.0479	.0433	.0393	.0358	.0328	.0302

TABLE 12

PART 3

m \ c	2.8	2.9	3.0	4.0	5.0	6.0	7.0
0	.3324	.3218	.3118	.2379	.1923	.1613	.1390
2	.1086	.1018	.0957	.0559	.0366	.0258	.0192
4	.0669	.0622	.0580	.0318	.0199	.0136	.0099
6	.0515	.0478	.0444	.0239	.0149	.0101	.0073
8	.0436	.0403	.0374	.0201	.0124	.0085	.0061
10	.0386	.0357	.0332	.0177	.0110	.0075	.0054
12	.0352	.0326	.0302	.0161	.0100	.0068	.0049
14	.0327	.0302	.0280	.0149	.0093	.0063	.0045
16	.0307	.0284	.0263	.0140	.0087	.0059	.0043
18	.0291	.0269	.0250	.0133	.0082	.0056	.0041
20	.0278	.0257	.0239	.0127	.0079	.0054	.0039

m \ c	8.0	9.0	10.0	11.0
0	.1220	.1088	.0981	.0893
2	.0148	.0118	.0096	.0079
4	.0075	.0059	.0047	.0039
6	.0055	.0043	.0035	.0029
8	.0046	.0036	.0029	.0024
10	.0041	.0032	.0026	.0021
12	.0037	.0029	.0023	.0019
14	.0034	.0027	.0022	.0018
16	.0032	.0025	.0020	.0017
18	.0031	.0024	.0019	.0016
20	.0029	.0023	.0018	.0015

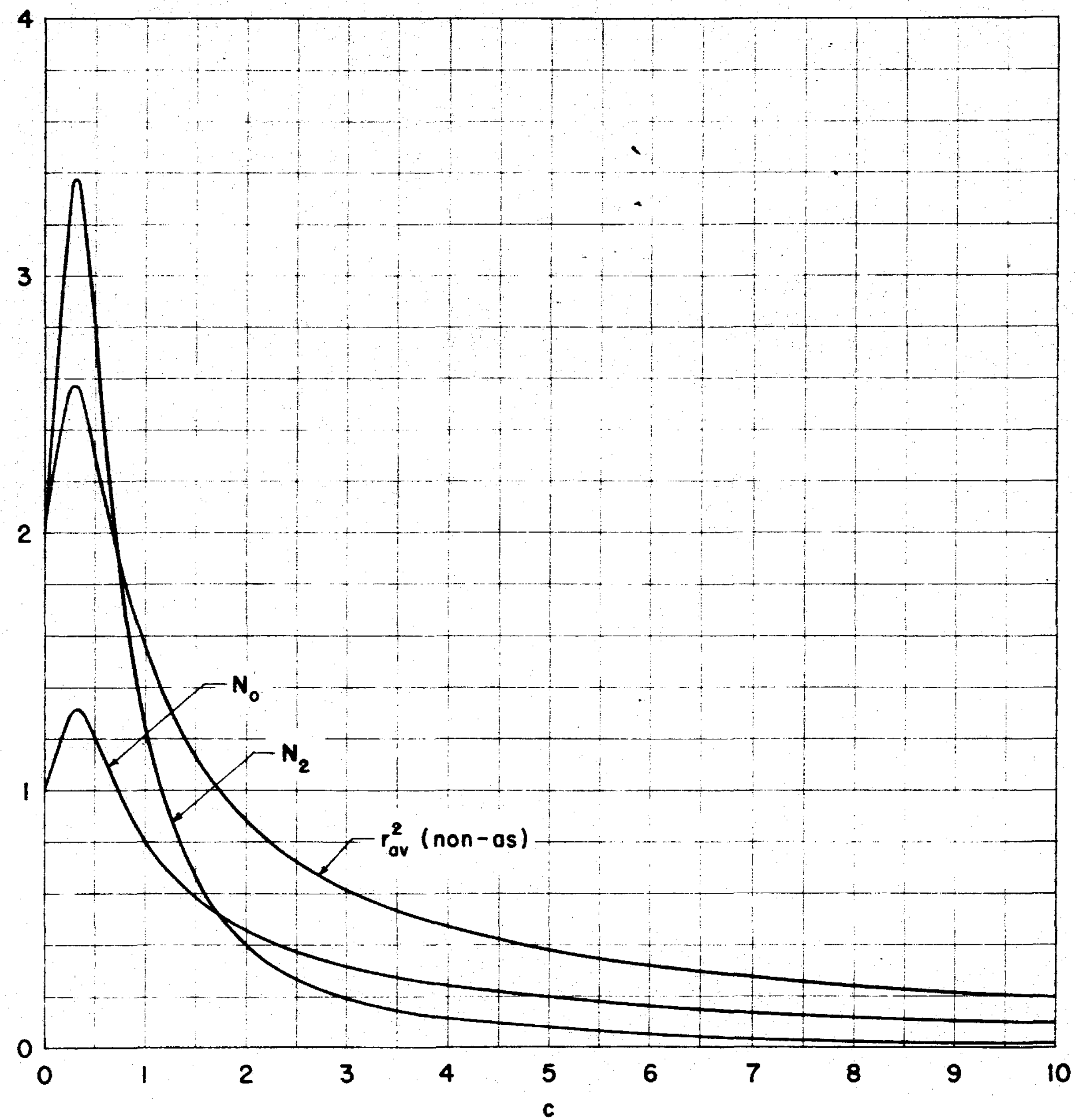


Fig. 20
POINT SOURCE IN INFINITE MEDIUM

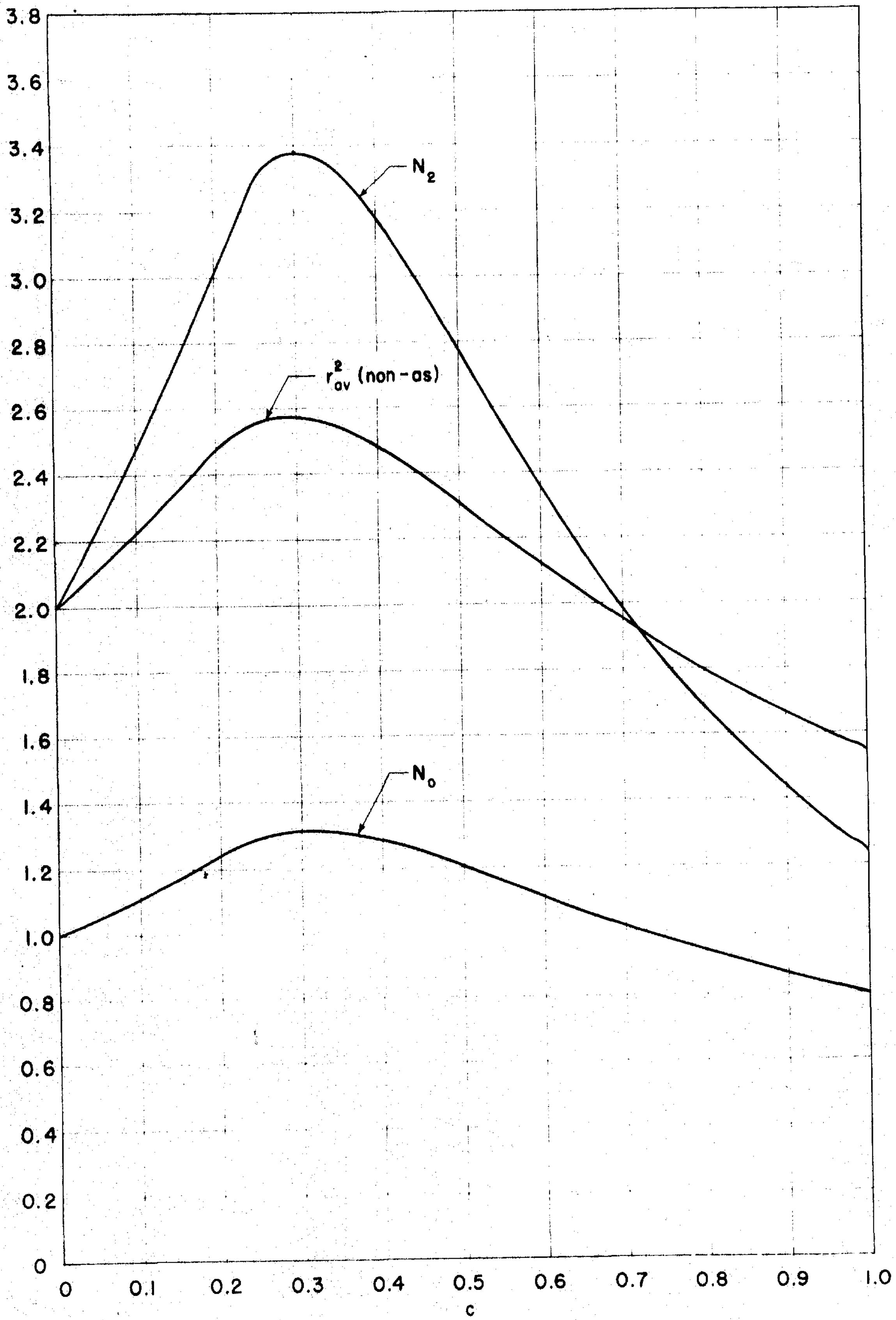


Fig. 21
POINT SOURCE IN INFINITE MEDIUM

TABLE 13

POINT SOURCE IN INFINITE MEDIUM

r^2_{av} (total) $= \frac{2}{1-c}$

r^2_{av} (asymptotic) $= 6/\kappa_0^2$

r^2_{av} (non-asymptotic) $= \frac{2}{1-c}\left\{\frac{1-3DK}{1-K}\right\}$

c	r^2_{av} (total)	r^2_{av} (asymptotic)	r^2_{av} (non-asymptotic)
0	2.000 000	6.000 000	2.0000
.1	2.222 222	6.000 000	2.2222
.2	2.500 000	6.001 091	2.4743
.3	2.857 143	6.031 155	2.5745
.4	3.333 333	6.176 306	2.4815
.5	4.000 000	6.544 403	2.3097
.6	5.000 000	7.288 170	2.1254
.7	6.666 667	8.738 259	1.9527
.8	10.000 000	11.888 606	1.7982
.9	20.000 000	21.733 132	1.6622
.92	25.000 000	26.704 866	1.6370
.94	33.333 333	35.010 787	1.6126
.96	50.000 000	51.650 858	1.5889
.98	100.000 000	101.625 052	1.5656
.99	200.000 000	201.612 408	1.5622
1.00	∞	∞	1.5429

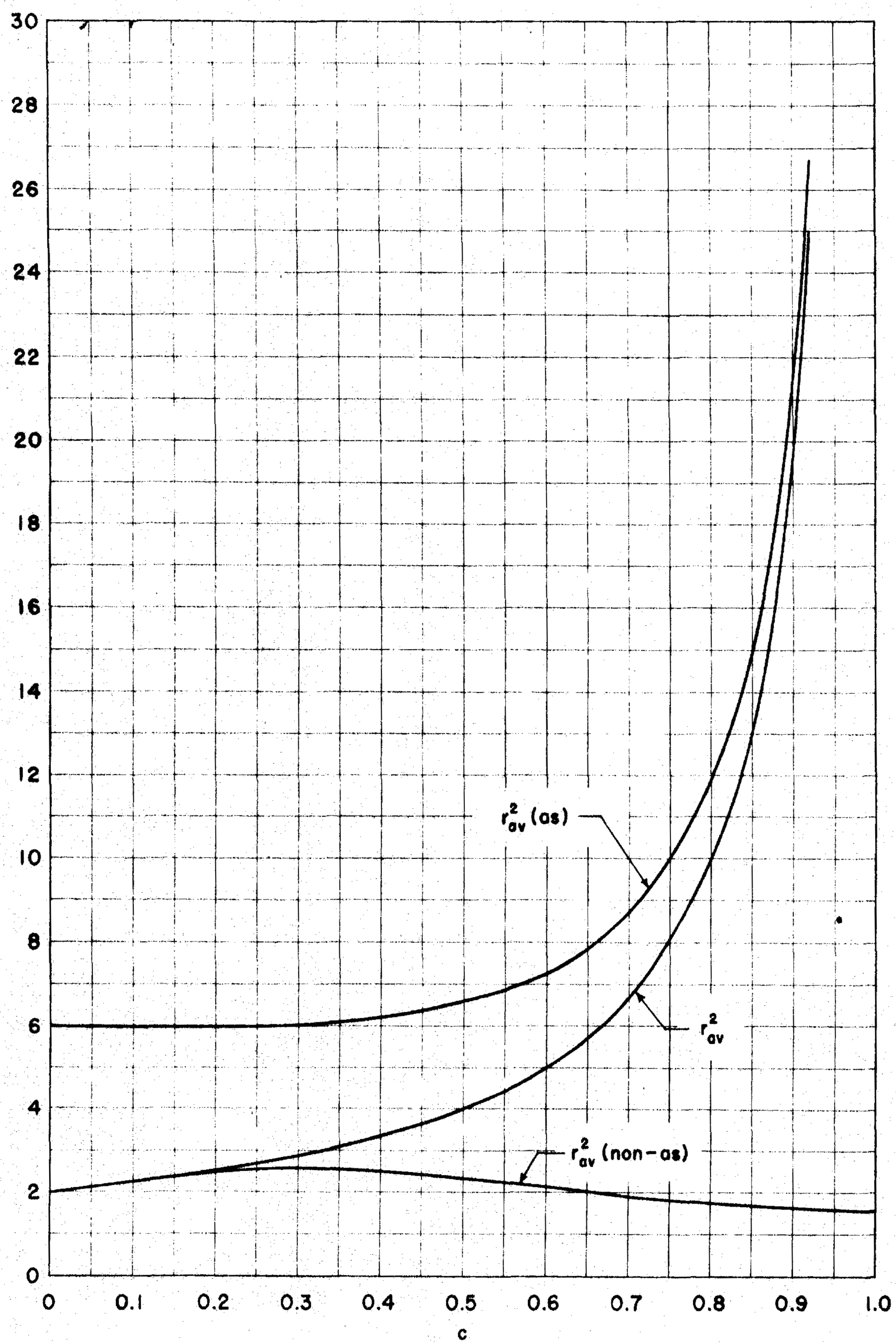


Fig. 22
POINT SOURCE IN INFINITE MEDIUM

in the integrand of many more complicated neutron problems.

From (31) we see that

$$g\ (c,\ 0) = g\ (0, \mu) = 1 \tag{54}$$

If $\mu \sim 1$ we find

$$g(c,\mu) \sim \frac{4}{c^2} \frac{1}{\left[\log \frac{(1-\mu)}{2}\right]^2} \tag{55}$$

and hence

$$g\ (c,1) = 0 \tag{56}$$

For $c > \frac{8}{\pi^2}$ g decreases monotonically from its value 1 at $\mu = 0$ to the value zero at $\mu = 1$. For $c < \frac{8}{\pi^2}$ g has a maximum in the interval between 0 and 1. As c decreases the maximum moves to the right and becomes steeper. For c very small

$$g_{max} \sim \frac{4}{\pi^2 c^2} \tag{57}$$

A useful form for g is obtained by expanding in powers μ^2 giving:

$$g\ (c,\mu) = 1 + \sum_{n=1}^{\infty} \Gamma_n \mu^{2n} \tag{58}$$

Table 14 gives the coefficients $\Gamma_n(c)$ from n = 1 to n = 5.

(58) is particularly suitable for use in integrands involving g (c,μ) since the integrals can then often be performed analytically*. Thus inserting (58) in (38) gives as an expansion for the non-asymptotic density p(r)

$$\begin{aligned} p(r) &= \frac{1}{4\pi r} \int_0^1 \left\{1 + \sum_{n=1}^{\infty} \Gamma_n \mu^{2n}\right\} e^{-r/\mu} \frac{d\mu}{\mu^2} \\ &= \frac{e^{-r}}{4\pi r^2} + \frac{1}{4\pi r} \sum_{n=1}^{\infty} \Gamma_n(c)\, E_{2n}(r) \end{aligned} \tag{59}$$

---

* "A table of integrals involving the functions $E_n(x)$" by J. LeCaine, Canadian Declassified Report MT 131 (NRC-1553), 1947.

TABLE 14

Γ_n (c)

c	m = 1	m = 2	m = 3	m = 4	m = 5
0.0	.000 0000	.000 0000	.000 0000	.000 0000	.000 0000
0.1	.175 3260	.087 4059	.058 5930	.044 5304	.036 1861
0.2	.301 3040	.184 1174	.136 9303	.111 2099	.094 8477
0.3	.377 9339	.252 8340	.197 1273	.164 7031	.143 0627
0.4	.405 2158	.270 8665	.206 3158	.166 6143	.138 9820
0.5	.383 1497	.230 1370	.153 4394	.105 8194	.073 1294
0.6	.311 7356	.137 1791	.055 2330	.010 1338	-.016 0744
0.7	.190 9735	.013 1375	-.048 6138	-.068 9470	-.072 0571
0.8	.020 8633	-.106 2314	-.111 1084	-.091 7518	-.071 1921
0.9	-.198 5949	-.170 5601	-.104 4227	-.064 5546	-.042 8277
1.0	-.467 4011	-.114 8695	-.057 1761	-.035 7425	-.025 0780

Multiplying (59) by $4\pi r^2 e^r$ gives as the corresponding expansion for $\epsilon(r)$:

$$\epsilon(r) = 1 + re^r \sum_{n=1}^{\infty} \Gamma_n(c) E_{2n}(r) \tag{60}$$

(60) may be used to investigate the behavior of $\epsilon(r)$ for r small. However, this may be found more simply using (28). Thus from (36), (37), and (39) we obtain

$$\epsilon(r) = 4\pi r^2 e^r \left\{ \rho(r) + \frac{\partial K_o^2}{\partial c} \frac{e^{-K_o r}}{4\pi r} \right\} \tag{61}$$

Inserting the expansion (28) for $\rho(r)$ we find for r small

$$\epsilon(r) \sim \left\{ 1 + \left(\frac{c\pi^2}{4} + \frac{\partial K_o^2}{\partial c}\right) r + c\left(2 - \frac{c\pi^2}{4}\right) r^2 \log r \right\} \tag{62}$$

Hence:

$$\epsilon(0) = 1 \tag{63}$$

$$\epsilon'(0) = \frac{c\pi^2}{4} + \frac{\partial K_o^2}{\partial c} \tag{64}$$

The expansion (62) may also be used to determine other properties of the function $g(c,\mu)$. Thus, for example, from (40) we have

$$\begin{aligned} \epsilon(r) &= re^r \int_0^1 \frac{g(c,\mu)}{\mu^2} e^{-r/\mu}\, du \\ &= 1 - re^r \int_0^1 \frac{1 - g(c,\mu)}{\mu^2} e^{-r/\mu}\, du \end{aligned} \tag{65}$$

Therefore:

$$-\int_0^1 \frac{1 - g(c,\mu)}{\mu^2} = \epsilon'(0) = \frac{c\pi^2}{4} + \frac{\partial K_o^2}{\partial c} \tag{66}$$

To investigate $\epsilon'(0)$ further, we note that

$$\text{For } c \ll 1, \quad \frac{\partial K_o^2}{\partial c} \approx \frac{8}{c^2} e^{-2/c} \quad \text{(from (13-80))} \tag{67}$$

$$\therefore \epsilon'(0) \approx \frac{c\pi^2}{4} \qquad c \ll 1 \tag{68}$$

$$\text{For } c \approx 1, \quad \frac{\partial K_o^2}{\partial c} \approx -3 \qquad \text{(from (13-81))} \tag{69}$$

$$\therefore \epsilon'(0) \approx \frac{\pi^2}{4} - 3 < 0 \qquad c \approx 1 \tag{70}$$

$$\text{For } c \gg 1, \quad \frac{\partial K_0^2}{\partial c} \approx -\frac{\pi^2}{2} c \qquad \text{(from (13-82))} \tag{71}$$

$$\therefore \epsilon'(0) \approx -\frac{\pi^2}{4} c \qquad c \gg 1 \tag{72}$$

Thus we see the slope of $\epsilon(r)$ at the origin changes sign as c increases. Indeed, using Table 8 and the expansions (13-80 to 13-82) one finds that starting from the value 0 at $c = 0$, $\epsilon'(0)$ rises to a maximum of 0.63589 at $c = 0.3404$, then decreases becoming negative at $c = 0.7647$. For c large $\epsilon'(0)$ is asymptotically a linear function of c.

From (62) we see that the coefficient of $r^2 \log r$ in the expansion of $\epsilon(r)$ also changes sign as c increases. For $c < \frac{8}{\pi^2}$ it is positive. For $c > \frac{8}{\pi^2}$ it is negative.

It may be noted that the equation for $\epsilon(r)$ in the vicinity of the origin (62) is actually more complicated than that for the total density (28). However, it is advantageous to use $\epsilon(r)$ since it is more convenient in later work.

The behavior of $\epsilon(r)$ in the opposite limit of $r \gg 1$ may also be obtained rather simply. Making the substitution

$$\mu = \frac{1}{1+x} \tag{73}$$

in (40) yields for arbitrary c and r

$$\epsilon(c,r) = r \int_0^\infty g\left(c, \frac{1}{1+x}\right) e^{-rx}\, dx \tag{74}$$

For $r \gg 1$ only, the region $x \ll 1$ will contribute appreciably to (74). In this region $g\left(c, \frac{1}{1+x}\right)$ can be approximated by

$$\bar{g}\,(c,x) = \frac{1}{\left(1 - \frac{c}{2}\log\frac{2}{x}\right)^2 + \left(\frac{\pi}{2}c\right)^2}$$

$$= \left(\frac{2}{\pi c}\right)^2 \frac{1}{1 + \left(\frac{1}{\pi}\log\frac{x}{x_o}\right)^2} \tag{75}$$

where $x_o = 2\,e^{-2/c}$ (76)

Introducing (76) into (74) and substituting

$$y = r\,x \tag{77}$$

yields

$$\epsilon\,(c,r) = \left(\frac{2}{\pi c}\right)^2 f\,(\lambda) \tag{78}$$

where

$$f\,(\lambda) = \int_0^\infty \frac{e^{-y}\,dy}{1 + \left(\frac{\lambda - \log y}{\pi}\right)^2} \tag{79}$$

and $\lambda = \log 2r - 2/c$. (80)

With the introduction of the variable λ it is apparent that in the limit of large r the function ϵ of the two variables c and r becomes a function only of the combination given by (80).

From (79) it is seen that $f(\lambda)$ is a rather simple function. It has a flat maximum of magnitude slightly less than one for some small negative value f λ. For large $|\lambda|$ it goes over into $\left(\frac{\pi}{\lambda}\right)^2$. $f(\lambda)$ is given numerically in Table 15 and graphed in Figures 23 and 24.

For $|\lambda| \ll \pi$ $f\,(\lambda)$ may be approximated by the Taylor series:

$$f\,(\lambda) = f(0) + \lambda f'(0) + \frac{\lambda^2}{2} f''(0) + \ldots \tag{81}$$

With repeated use of the relation

TABLE 15

THE FUNCTION f(λ)

λ	f(λ)	$\left(\frac{\lambda}{\pi}\right)^2 f(\lambda)$
-20	.0259	1.048
-18	.0320	1.052
-16	.0407	1.057
-14	.0535	1.062
-12	.0732	1.068
-10	.1058	1.072
- 8	.1640	1.063
- 5	.3624	.918
- 3	.6208	.566
- 2	.7670	.311
- 1	.8723	.088
- .5	.8920	.023
0	.8832	.000
.5	.8460	.021
1	.7851	.080
2	.6256	.254
3	.4686	.427
5	.2600	.659
8	.1236	.801
10	.0835	.846
12	.0601	.877
14	.0452	.898
16	.0352	.913
18	.0281	.924
20	.0230	.933

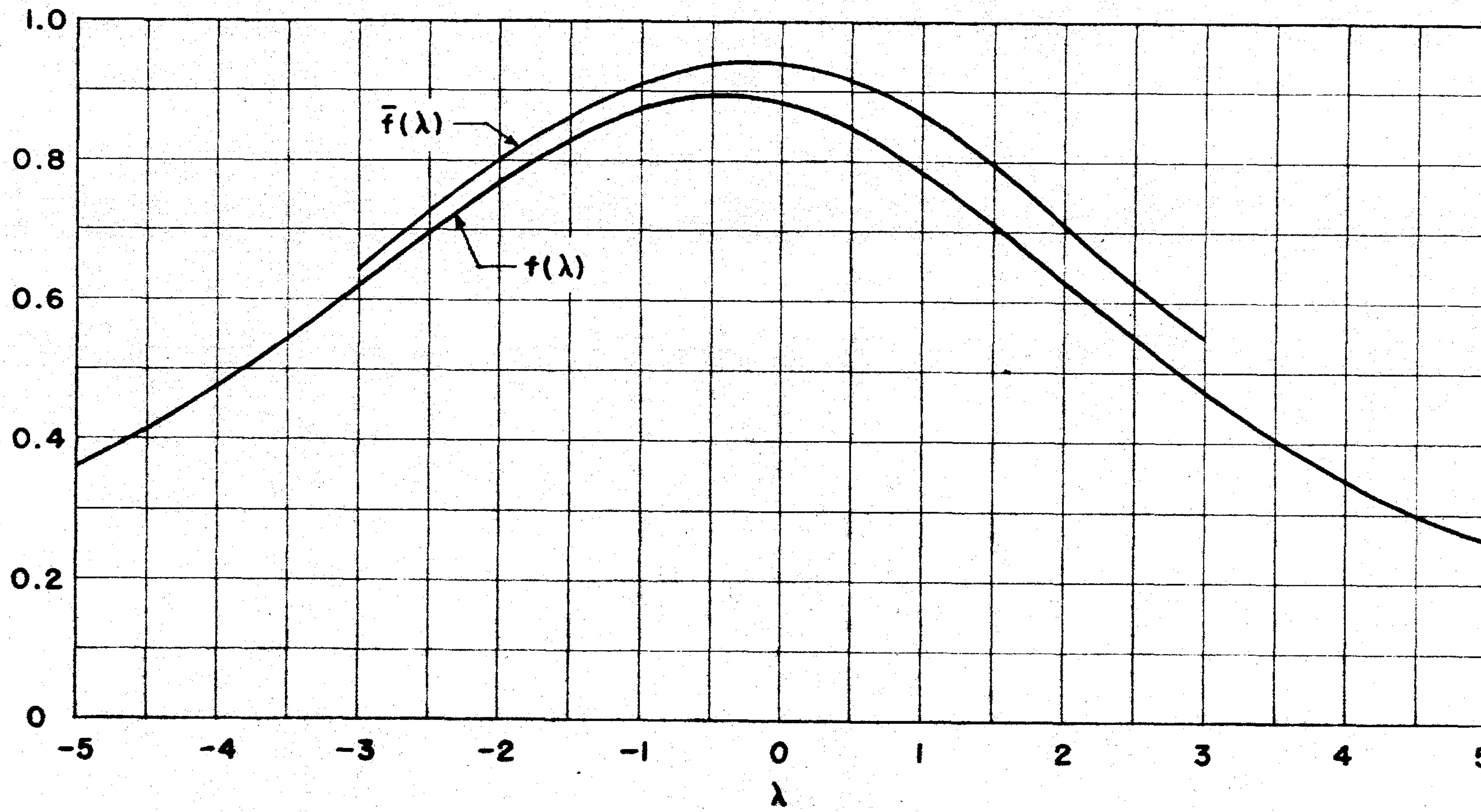


Fig. 23

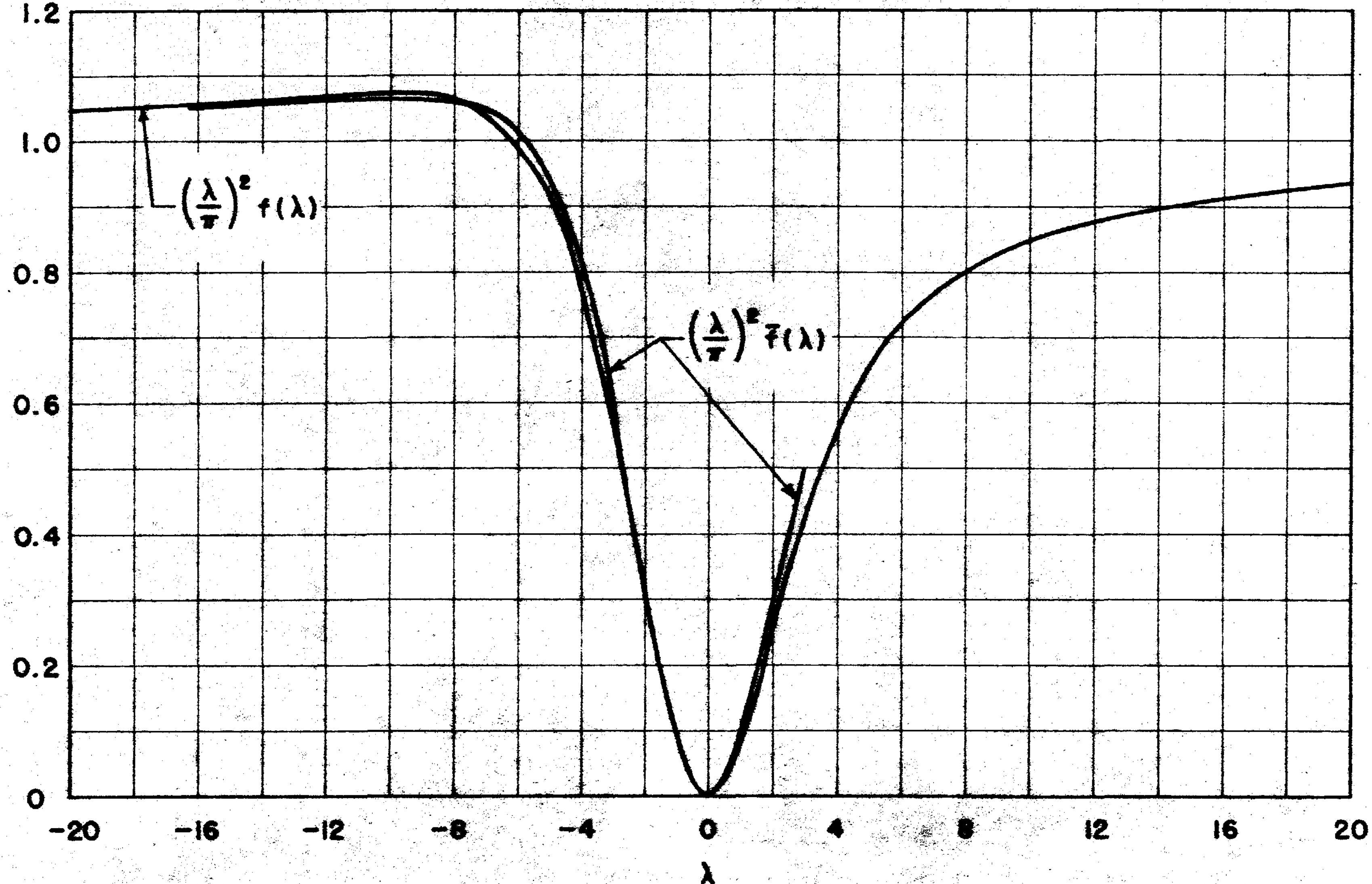

1.2
1.0
0.8
0.6
0.4
0.2
0
−20
−16
−12
−8
−4
0
4
8
12
16
20
λ
$\left(\frac{\lambda}{\pi}\right)^2 f(\lambda)$
$\left(\frac{\lambda}{\pi}\right)^2 \bar{f}(\lambda)$

Fig. 24

$$\frac{d}{d\lambda}\frac{1}{1+\left(\frac{\lambda-\log y}{\pi}\right)^2}=-\frac{d}{d\log y}\frac{1}{1+\left(\frac{\lambda-\log y}{\pi}\right)^2}$$

$$=-y\frac{d}{dy}\frac{1}{1+\left(\frac{\lambda-\log y}{\pi}\right)^2} \qquad (82)$$

and integrating by parts one finds

$$f(0)=\int_0^\infty\frac{e^{-y}\,dy}{1+\left(\frac{\log y}{\pi}\right)^2}=0.88322 \qquad (83)$$

$$f'(0)=-\int_0^\infty\frac{1-y}{1+\left(\frac{\log y}{\pi}\right)^2}e^{-y}\,dy=-0.04684 \qquad (84)$$

$$f''(0)=-\int_0^\infty\frac{3y-1-y^2}{1+\left(\frac{\log y}{\pi}\right)^2}e^{-y}\,dy=-0.11500 \qquad (85)$$

Similarly $f'''(0)=0.01863$, $f^{IV}(0)=0.08280$ and $f^{V}(0)=-0.0202$.

Using the first six terms in (81) gives $f(\lambda)$ correct to four significant figures for $|\lambda|\lesssim 0.5$.

For large $|\lambda|$ (i.e., $\lambda^2\gg\pi^2$) an approximate expression for $f(\lambda)$ may be found by expanding the integrand of (79) in powers of $1/\lambda$ and integrating term by term. Of course in part of the region of the integration the expansion of the integrand will not be convergent. However, it can be shown that for large $|\lambda|$ the contribution of this region is negligible. In this way one finds

$$f(\lambda)=\left(\frac{\pi}{\lambda}\right)^2\left\{1-\frac{2\gamma}{\lambda}-\frac{\pi^2/2-3\gamma^2}{\lambda^2}+0\left(\frac{1}{\lambda^3}\right)\right. \qquad (86)$$

where

$$\left.\begin{array}{r}\lambda^2\gg\pi^2\\ \gamma=0.5772\ \ldots\end{array}\right\} \qquad (87)$$

The above properties of $f(\lambda)$ allow us now to discuss the asymptotic behavior of $\epsilon(c,r)$. We have, of course, to keep in mind that large negative values of λ are compatible with the condition $r\gg1$ only if $c\ll1$. Small values of $|\lambda|$ are only possible if $c\lesssim 0.4$. For larger c the maximum of $\epsilon(r)$ (if it exists at all) will be at distances which are not large with respect to 1 and hence cannot be treated by asymptotic methods. In other words: For $c \ll 1$ the asymptotic formulae will cover practically the whole range of $\epsilon(r)$ starting from $\epsilon(r) = 1 + O(c)$ for $\log r \ll 2/c$ via the maximum for large r and the subsequent decrease for still larger r. For c = 0.4 or 0.5 only the region to the right of the maximum is asymptotically describable. For $c \gg 1$ the asymptotic description will only hold when $\epsilon(r)$ is already small compared to one.

The expressions for $\epsilon(r)$ for large r in the various limiting cases following from the above expansions for $f(\lambda)$ are as follows:

<u>$\log r \ll 2/c$</u> $\qquad (\therefore c \ll 1)$

$$\epsilon(c,r) = 1 + c \left\{ (\log r) + 1.2704 \right\} + \dots \tag{88}$$

<u>$\log 2r \cong 2/c$</u> $\qquad (\therefore c \lesssim 0.4)$

$$\epsilon(c,r) = \frac{0.358}{c^2} \left\{ 1 - 0.05303 \left[(\log 2r) - \frac{2}{c} \right] - \dots \right\} \tag{89}$$

In this case we see that the maximum of $\epsilon(r)$ is

$$\epsilon_{max} \approx \frac{0.36}{c^2} \tag{90}$$

which occurs at

$$r = \frac{1}{2} e^{2/c + \lambda_o} \tag{91}$$

where

$$\lambda_o \cong - 0.4 \tag{92}$$

<u>$(\log 2r - \frac{2}{c})^2 \gg \pi^2$</u>

This includes all c for sufficiently large r. "Sufficiently large r" means

$$\left.\begin{array}{ll} \log r \gg 2/c & c \ll 1 \\ (\log r)^2 \gg \pi^2 & c > 1 \end{array}\right\} \tag{93}$$

$$\epsilon(c,r) = \left(\frac{2}{c \log r} \right)^2 \left\{ 1 + \frac{4/c - 2.54}{\log r} + \dots \right\} \tag{94}$$

The ultimate asymptotic behavior for all c is thus

$$\epsilon(c,r) \sim \frac{4}{c^2(\log r)^2}$$

From Table 16 for ϵ the function

$$\bar{f}(\lambda) = (\frac{\pi c}{2})^2 \epsilon \tag{95}$$

can be constructed. Plotting this as a function of λ one obtains, for $r = 20$, reasonably good agreement with $f(\lambda)$. From Figures 23 and 24 we see that $\bar{f}$ lies slightly higher than f. For r = 10, the deviation of $\bar{f}(\lambda)$ from $f(\lambda)$ is already much more pronounced. This is, of course, just what is to be expected.

Using our present expansions we can now investigate the point where the asymptotic neutron density begins to dominate the non-asymptotic density. From (39) and (37) we have

$$\frac{p(r)}{\rho_{as}(r)} = \left(\frac{-\partial K_o^2}{\partial c}\right)^{-1} \frac{e^{-(1-K_o)r}}{r} \epsilon(r) \tag{96}$$

Expressing $\epsilon(r)$ in terms of $f(\lambda)$ by means of (78) and using the small c expressing for K_o and $\frac{\partial K_o^2}{\partial c}$ (approximately valid up to 0.4), we obtain

$$\frac{p(r)}{\rho_{as}(r)} = \frac{e^{-\rho}}{\rho} \frac{f(\log \rho)}{\pi^2} \tag{97}$$

where

$$\rho = e^{\lambda} = 2re^{-2/c} \tag{98}$$

The condition that $p(r) = \rho_{as}(r)$ is then

$$f(\log \rho) = \pi^2 \rho e^{\rho} \tag{99}$$

Here e^{ρ} may be put equal to one without appreciable error. (99) then becomes

$$f(\lambda) = \pi^2 e^{\lambda} \tag{100}$$

This is solved by

$$\lambda \approx -2.78 \tag{101}$$

so that

$$r = 0.031\, e^{2/c} \tag{102}$$

For intermediate values of r, $\epsilon(r)$ must be obtained from (40) by numerical integration. Table 16 gives values of ϵ as a function of r determined in this way. ϵ is graphed in Figure 25. From this figure it is seen that the properties of g (c, μ) are mirrored in the behavior of ϵ (r). Thus for c sufficiently large ($c \gtrsim 8/\pi^2$) ϵ decreases monotonically with c. For smaller c ϵ has a maximum. As c decreases further the maximum moves out to larger r and becomes higher. As a consequence the curves of ϵ (r) for different c cross. It should be noted that since ϵ occurs multiplied by $e^{-r}/4\pi r^2$ the flat maximum of ϵ will not be important for the density ρ.

Now that we have ϵ (r) it is quite trivial to obtain the total density. Thus, by (39), we have p(r) on multiplying ϵ by $e^{-r}/4\pi r^2$. Adding the asymptotic density gives ρ in accordance with (36). Tables 17 and 18 give the resulting values of $4\pi r^2 \rho_{as}$, $4\pi r^2$ p and $4\pi r^2 \rho$ for c = 0.3 and 0.9. These are plotted in Figures 26 and 27. The ratio ρ_{as}/ρ as a function of r is given in Table 19. This is plotted in Figures 28 and 29.

Comparing Figures 26 and 27 we note that the behavior of the density in the vicinity of the origin varies considerably with c. For c = 0.3 $4\pi r^2 \rho$ decreases monotonically. On the other hand, when c = 0.9, $4\pi r^2 \rho$ first increases to a maximum and then decreases. Also we see that, although almost the whole density is the asymptotic density after 3 mean free paths for c = 0.9, for c = 0.3 practically all neutrons are in the non-asymptotic density. Further light on this is found in Figures 28 and 29 from which we find

<u>For c = 1</u>	at 1 m.f.p.	$p/\rho \sim 8.5$ %
	at 2.5 m.f.p.	$p/\rho \sim 0.7$ %

r \ c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	1.0000	1.0210	1.0418	1.0542	1.0526	1.0420	1.0269	1.0099	.9921	.9745	.9564
.2	1.0000	1.0382	1.0773	1.1000	1.0962	1.0756	1.0474	1.0162	.9843	.9528	.9222
.3	1.0000	1.0532	1.1088	1.1409	1.1346	1.1046	1.0643	1.0206	.9767	.9341	.8934
.4	1.0000	1.0667	1.1375	1.1781	1.1692	1.1301	1.0786	1.0236	.9693	.9173	.8683
.5	1.0000	1.0790	1.1640	1.2126	1.2008	1.1529	1.0909	1.0257	.9621	.9019	.8460
.6	1.0000	1.0904	1.1888	1.2448	1.2300	1.1736	1.1017	1.9271	.9551	.8878	.8260
.7	1.0000	1.1010	1.2121	1.2752	1.2571	1.1926	1.1113	1.0279	.9483	.8747	.8077
.8	1.0000	1.1109	1.2342	1.3038	1.2826	1.2100	1.1198	1.0282	.9417	.8625	.7910
.9	1.0000	1.1202	1.2552	1.3311	1.3066	1.2262	1.1275	1.9282	.9353	.8510	.7755
1.0	1.0000	1.1291	1.2753	1.3571	1.3293	1.2412	1.1343	1.0278	.9290	.8402	.7612
1.5	1.0000	1.1674	1.3644	1.4724	1.4273	1.3034	1.1601	1.0229	.9002	.7936	.7019
2.0	1.0000	1.1990	1.4402	1.5699	1.5068	1.3504	1.1763	1.0149	.8748	.7562	.6568
2.5	1.0000	1.2258	1.5068	1.6551	1.5738	1.3874	1.1866	1.0054	.8519	.7250	.6207
3.0	1.0000	1.2494	1.5667	1.7311	1.6314	1.4171	1.1929	.9952	.8313	.6982	.5908
3.5	1.0000	1.2704	1.6213	1.8000	1.6818	1.4415	1.1963	.9847	.8124	.6749	.5655
4.0	1.0000	1.2895	1.6718	1.8630	1.7265	1.4617	1.1978	.9742	.7951	.6543	.5437
4.5	1.0000	1.3070	1.7188	1.9214	1.7665	1.4786	1.1976	.9637	.7791	.6358	.5246
5.0	1.0000	1.3231	1.7630	1.9757	1.8026	1.4928	1.1963	.9534	.7643	.6191	.5076
6.0	1.0000	1.3521	1.8443	2.0745	1.8654	1.5147	1.1912	.9334	.7374	.5901	.4788
7.0	1.0000	1.3779	1.9182	2.1630	1.9182	1.5304	1.1838	.9144	.7137	.5654	.4550
8.0	1.0000	1.4010	1.9863	2.2432	1.9634	1.5412	1.1749	.8964	.6926	.5440	.4349
9.0	1.0000	1.4222	2.0497	2.3169	2.0024	1.5486	1.1651	.8793	.6734	.5253	.4175
10.0	1.0000	1.4417	2.1094	2.3851	2.0366	1.5531	1.1547	.8631	.6560	.5086	.4024
11.0	1.0000	1.4599	2.1659	2.4499	2.0667	1.5554	1.1438	.8477	.6400	.4936	.3890
12.0	1.0000	1.4770	2.2196	2.5086	2.0933	1.5559	1.1327	.8330	.6252	.4800	.3769
13.0	1.0000	1.4931	2.2710	2.5652	2.1172	1.5550	1.1215	.8190	.6114	.4676	.3661
14.0	1.0000	1.5084	2.3204	2.6188	2.1385	1.5529	1.1102	.8055	.5985	.4562	.3562
15.0	1.0000	1.5230	2.3682	2.6700	2.1578	1.5498	1.0989	.7926	.5864	.4456	.3471
16.0	1.0000	1.5370	2.4141	2.7190	2.1752	1.5459	1.0876	.7802	.5750	.4357	.3387
17.0	1.0000	1.5503	2.4586	2.7658	2.1910	1.5413	1.0764	.7683	.5643	.4265	.3310
18.0	1.0000	1.5632	2.5019	2.8109	2.2055	1.5361	1.0653	.7568	.5540	.4178	.3238
19.0	1.0000	1.5757	2.5439	2.8543	2.2186	1.5304	1.0542	.7457	.5443	.4096	.3170
20.0	1.0000	1.5877	2.5849	2.8963	2.2307	1.5243	1.0433	.7349	.5349	.4019	.3107

TABLE 16 (SHEET 1)

$\mathcal{E}$ (r) For Point Source

0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	.9564	.9389	.9218	.9052	.8890	.8733	.8580	.8431	.8287	.8146	.8010	.7877	.7748	.7623	.7500
.2	.9222	.8928	.8646	.8377	.8119	.7874	.7639	.7415	.7201	.6997	.6801	.6613	.6434	.6262	.6096
.3	.8934	.8549	.8186	.7844	.7522	.7219	.6933	.6664	.6410	.6170	.5944	.5729	.5526	.5334	.5151
.4	.8683	.8226	.7800	.7404	.7036	.6693	.6374	.6076	.5799	.5539	.5296	.5068	.4854	.4654	.4465
.5	.8460	.7944	.7469	.7031	.6629	.6258	.5916	.5601	.5308	.5038	.4786	.4553	.4335	.4132	.3943
.6	.8260	.7694	.7179	.6710	.6282	.5891	.5533	.5206	.4905	.4629	.4374	.4139	.3921	.3720	.3533
.7	.8077	.7471	.6923	.6428	.5980	.5575	.5207	.4872	.4567	.4288	.4033	.3799	.3584	.3385	.3202
.8	.7910	.7268	.6693	.6178	.5716	.5300	.4925	.4586	.4279	.4000	.3747	.3515	.3303	.3109	.2931
.9	.7755	.7084	.6486	.5955	.5481	.5057	.4678	.4338	.4031	.3754	.3502	.3274	.3067	.2878	.2704
1.0	.7612	.6914	.6298	.5753	.5271	.4842	.4461	.4120	.3814	.3539	.3292	.3068	.2865	.2680	.2512
1.5	.7019	.6233	.5559	.4979	.4478	.4044	.3666	.3335	.3044	.2788	.2561	.2359	.2179	.2018	.1873
2.0	.6568	.5736	.5037	.4448	.3949	.3524	.3159	.2845	.2573	.2336	.2129	.1947	.1786	.1643	.1516
2.5	.6207	.5350	.4644	.4057	.3567	.3155	.2806	.2509	.2254	.2034	.1844	.1678	.1532	.1404	.1290
3.0	.5908	.5040	.4333	.3754	.3276	.2879	.2545	.2263	.2023	.1818	.1641	.1488	.1355	.1238	.1135
3.5	.5655	.4782	.4081	.3512	.3047	.2663	.2343	.2075	.1849	.1656	.1490	.1348	.1224	.1116	.1021
4.0	.5437	.4564	.3870	.3312	.2860	.2489	.2182	.1926	.1711	.1529	.1373	.1239	.1123	.1022	.0934
4.5	.5246	.4377	.3691	.3145	.2704	.2345	.2050	.1805	.1600	.1426	.1279	.1152	.1043	.0948	.0865
5.0	.5076	.4213	.3537	.3001	.2572	.2225	.1940	.1704	.1508	.1342	.1202	.1081	.0978	.0888	.0810
6.0	.4788	.3939	.3284	.2769	.2360	.2033	.1766	.1546	.1364	.1211	.1082	.0972	.0878	.0796	.0725
7.0	.4550	.3717	.3081	.2586	.2196	.1885	.1633	.1427	.1256	.1114	.0993	.0891	.0804	.0728	.0663
8.0	.4349	.3535	.2915	.2438	.2065	.1768	.1528	.1333	.1172	.1038	.0925	.0829	.0747	.0676	.0615
9.0	.4175	.3378	.2777	.2316	.1956	.1672	.1443	.1257	.1104	.0977	.0870	.0779	.0702	.0635	.0578
10.0	.4024	.3243	.2658	.2212	.1865	.1591	.1372	.1194	.1048	.0926	.0824	.0738	.0665	.0601	.0547
11.0	.3890	.3125	.2555	.2122	.1787	.1523	.1312	.1140	.1000	.0884	.0786	.0704	.0634	.0573	.0521
12.0	.3769	.3020	.2465	.2044	.1719	.1463	.1259	.1095	.0959	.0848	.0754	.0675	.0607	.0549	.0499
13.0	.3661	.2927	.2384	.1974	.1659	.1411	.1214	.1055	.0924	.0816	.0726	.0649	.0584	.0529	.0481
14.0	.3562	.2842	.2312	.1913	.1605	.1365	.1174	.1019	.0893	.0788	.0701	.0627	.0565	.0511	.0464
15.0	.3471	.2765	.2247	.1857	.1558	.1324	.1138	.0988	.0865	.0764	.0679	.0608	.0547	.0495	.0450
16.0	.3387	.2695	.2187	.1806	.1514	.1286	.1105	.0959	.0840	.0742	.0660	.0590	.0531	.0481	.0437
17.0	.3310	.2630	.2133	.1760	.1475	.1253	.1076	.0934	.0818	.0723	.0643	.0575	.0518	.0469	.0426
18.0	.3238	.2570	.2082	.1718	.1439	.1222	.1050	.0911	.0798	.0705	.0627	.0561	.0505	.0457	.0416
19.0	.3170	.2514	.2036	.1679	.1406	.1194	.1026	.0890	.0780	.0689	.0613	.0549	.0494	.0448	.0408
20.0	.3107	.2462	.1992	.1643	.1376	.1168	.1004	.0871	.0763	.0675	.0600	.0538	.0484	.0439	.0400

$\mathcal{E}$ (r) For Point Source

TABLE 16 (Sheet 2)

0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	.7381	.7265	.7152	.7042	.6935	.6830
.2	.5938	.5785	.5639	.5498	.5362	.5232
.3	.4977	.4812	.4655	.4505	.4363	.4227
.4	.4287	.4119	.3960	.3811	.3669	.3535
.5	.3766	.3600	.3444	.3298	.3160	.3031
.6	.3359	.3197	.3046	.2905	.2773	.2650
.7	.3033	.2876	.2731	.2596	.2470	.2352
.8	.2767	.2616	.2476	.2346	.2226	.2115
.9	.2546	.2400	.2266	.2142	.2027	.1922
1.0	.2359	.2218	.2089	.1971	.1862	.1761
1.5	.1743	.1625	.1518	.1421	.1333	.1252
2.0	.1403	.1302	.1210	.1128	.1053	.0986
2.5	.1190	.1100	.1020	.0948	.0883	.0824
3.0	.1044	.0963	.0891	.0826	.0768	.0716
3.5	.0937	.0863	.0798	.0739	.0686	.0639
4.0	.0856	.0788	.0727	.0673	.0625	.0581
4.5	.0793	.0729	.0672	.0622	.0576	.0536
5.0	.0741	.0681	.0628	.0580	.0538	.0500
6.0	.0663	.0608	.0560	.0517	.0479	.0445
7.0	.0606	.0555	.0511	.0472	.0437	.0406
8.0	.0562	.0515	.0474	.0437	.0405	.0376
9.0	.0527	.0483	.0445	.0410	.0380	.0352
10.0	.0499	.0457	.0421	.0388	.0359	.0333
11.0	.0476	.0436	.0401	.0370	.0342	.0318
12.0	.0456	.0418	.0384	.0354	.0328	.0304
13.0	.0439	.0402	.0370	.0341	.0316	.0293
14.0	.0424	.0388	.0357	.0330	.0305	.0283
15.0	.0411	.0376	.0346	.0320	.0296	.0275
16.0	.0399	.0366	.0337	.0311	.0288	.0267
17.0	.0389	.0357	.0329	.0304	.0281	.0261
18.0	.0380	.0349	.0321	.0297	.0275	.0256
19.0	.0373	.0342	.0315	.0291	.0270	.0251
20.0	.0365	.0335	.0309	.0286	.0265	.0247

$\mathcal{E}$ (r) For Point Source

TABLE 16 (Sheet 3)

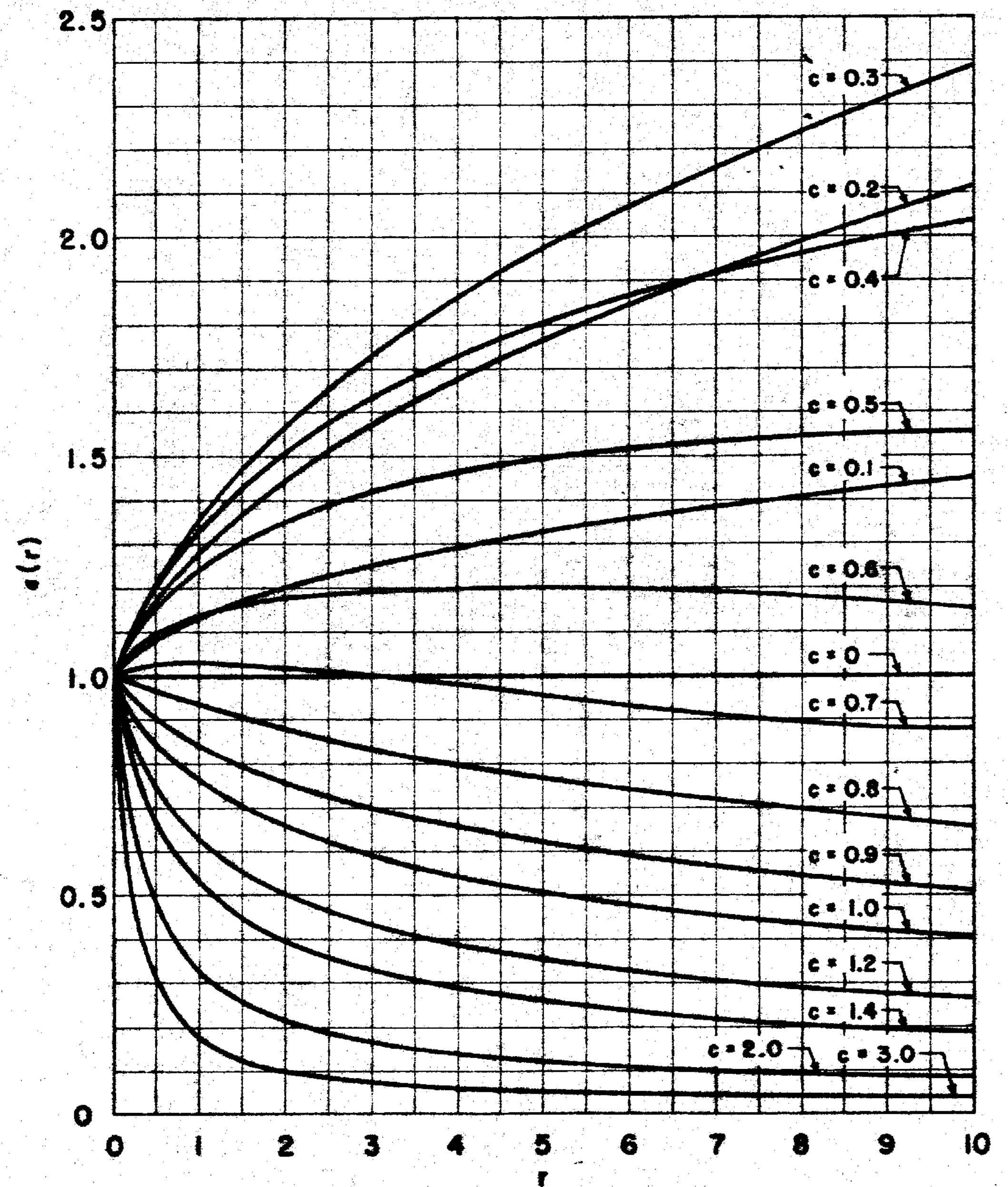


Fig. 25

POINT SOURCE IN INFINITE MEDIUM

r	$4\pi r^2 \rho_{as}$	$4\pi r^2 p$	$4\pi r^2 \rho$
0	.0000	1.0000	1.0000
.1	.0105	.9539	.9644
.2	.0190	.9006	.9196
.3	.0258	.8452	.8710
.4	.0312	.7897	.8209
.5	.0353	.7355	.7708
.6	.0383	.6832	.7215
.7	.0405	.6332	.6737
.8	.0419	.5858	.6277
.9	.0426	.5412	.5838
1.0	.0429	.4992	.5421
1.5	.0390	.3285	.3675
2.0	.0316	.2125	.2441
2.5	.0240	.1359	.1599
3.0	.0175	.0862	.1037
3.5	.0124	.0544	.0668
4.0	.0086	.0341	.0427
4.5	.0059	.0213	.0272
5.0	.0040	.0133	.0173
6.0	.0018	.0051	.0069
7.0	.0008	.0020	.0028
8.0	.0003	.0008	.0011
9.0	.0001	.0003	.0004
10.0	.0001	.0001	.0002

Point Source in Infinite Medium

TABLE 17

$c = .3$

r	$4\pi r^2 \rho_{as}$	$4\pi r^2 p$	$4\pi r^2 \rho$
0	0	1.0000	1.0000
.1	.2393	.8818	1.1211
.2	.4542	.7801	1.2343
.3	.6464	.6920	1.3384
.4	.8177	.6149	1.4326
.5	.9698	.5470	1.5168
.6	1.1042	.4872	1.5914
.7	1.2223	.4344	1.6567
.8	1.3254	.3876	1.7130
.9	1.4147	.3460	1.7607
1.0	1.4915	.3091	1.8006
1.5	1.7203	.1771	1.8974
2.0	1.7637	.1023	1.8660
2.5	1.6952	.0595	1.7547
3.0	1.5644	.0348	1.5992
3.5	1.4035	.0204	1.4239
4.0	1.2334	.0120	1.2454
4.5	1.0671	.0071	1.0742
5.0	.9116	.0042	.9158
6.0	.6468	.0015	.6483
7.0	.4462	.0005	.4467
8.0	.3016	.0002	.3018
9.0	.2006	.0001	.2007
10.0	.1318	.0000	.1318

Point Source in Infinite Medium

TABLE 18

$c = .9$

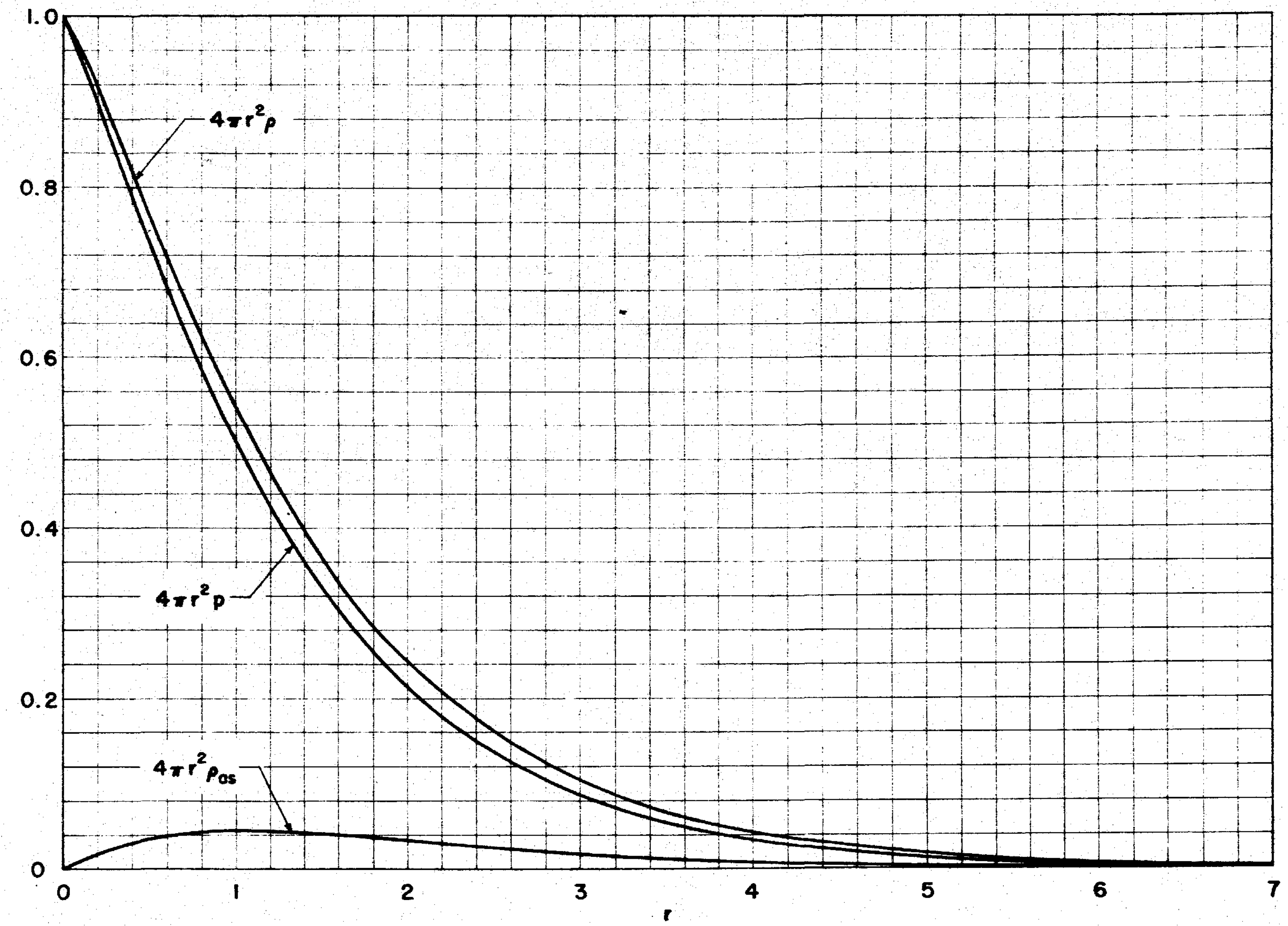


Fig. 26
POINT SOURCE IN INFINITE MEDIUM AND c = 0.3

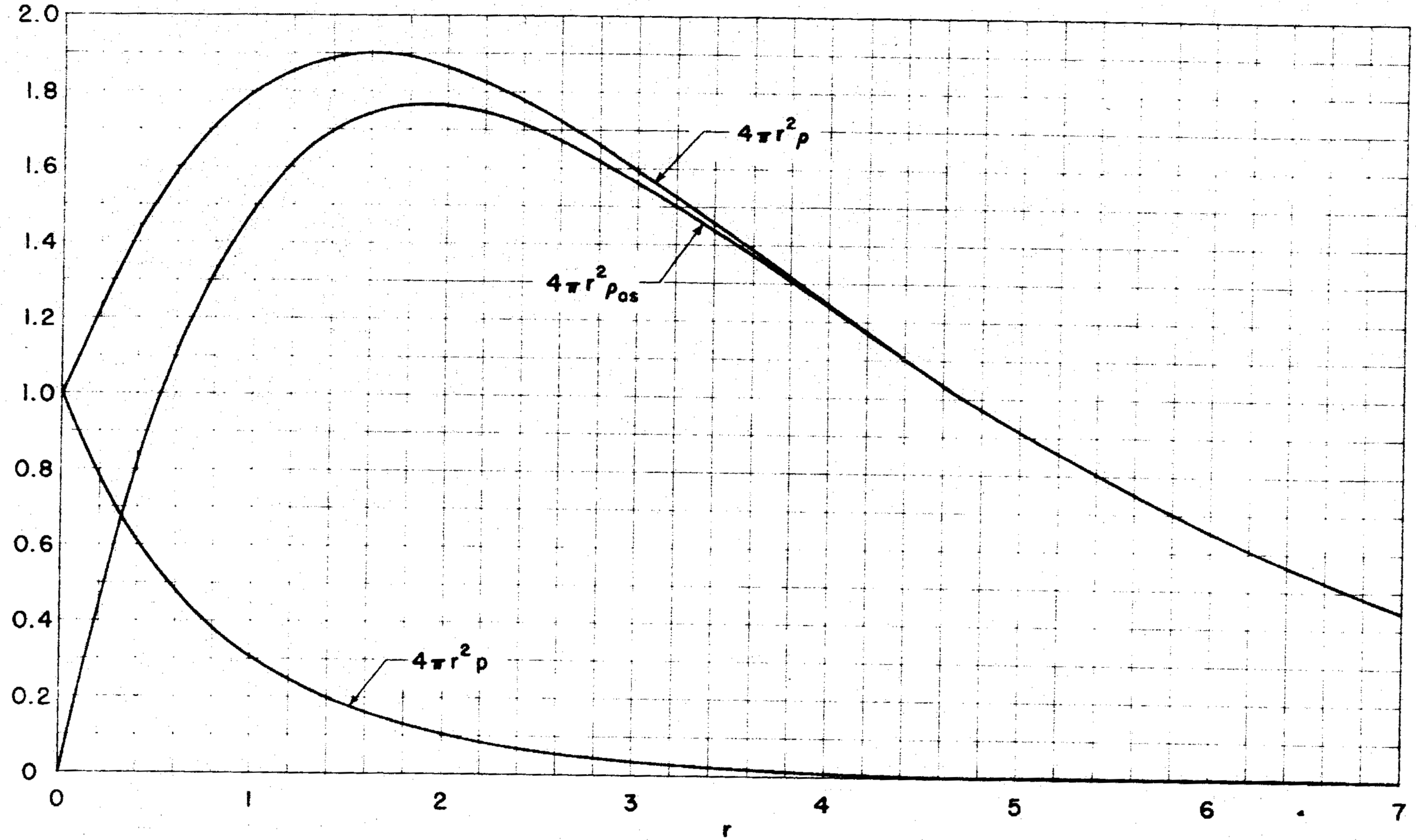


Fig. 27
POINT SOURCE IN INFINITE MEDIUM AND c = 0.9

c / $\tilde{r}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.1	.0000	1.615×10^{-7}	8.721×10^{-4}	1.090×10^{-2}	3.430×10^{-2}	6.589×10^{-1}	.1012	.1380	.1755	.2135	.2574
.2	.0000	3.176×10^{-7}	1.686×10^{-3}	2.070×10^{-2}	6.394×10^{-2}	.1207	.1823	.2446	.3063	.3680	.4428
.3	.0000	4.697×10^{-7}	2.455×10^{-3}	2.967×10^{-2}	9.019×10^{-2}	.1676	.2493	.3298	.4073	.4830	.5762
.4	.0000	6.183×10^{-7}	3.188×10^{-3}	3.799×10^{-2}	.1138	.2086	.3060	.3996	.4873	.5708	.6734
.5	.0000	7.641×10^{-7}	3.891×10^{-3}	4.578×10^{-2}	.1354	.2449	.3549	.4578	.5520	.6394	.7451
.6	.0000	9.073×10^{-7}	4.569×10^{-3}	5.312×10^{-2}	.1552	.2774	.3975	.5072	.6052	.6939	.7988
.7	.0000	1.048×10^{-6}	5.224×10^{-3}	6.006×10^{-2}	.1735	.3068	.4351	.5497	.6469	.7378	.8396
.8	.0000	1.188×10^{-6}	5.861×10^{-3}	6.668×10^{-2}	.1906	.3336	.4686	.5866	.6872	.7738	.8710
.9	.0000	1.325×10^{-6}	6.479×10^{-3}	7.300×10^{-2}	.2066	.3582	.4986	.6189	.7192	.8035	.8954
1.0	.0000	1.460×10^{-6}	7.081×10^{-3}	7.906×10^{-2}	.2217	.3809	.5257	.6474	.7468	.8284	.9147
1.5	.0000	2.119×10^{-6}	9.900×10^{-3}	.1062	.2861	.4731	.6300	.7509	.8407	.9067	.9664
2.0	.0000	2.750×10^{-6}	1.247×10^{-2}	.1295	.3377	.5413	.7011	.8153	.8933	.9452	.9854
2.5	.0000	3.363×10^{-6}	1.487×10^{-2}	.1501	.3807	.5946	.7527	.8585	.9255	.9661	.9932
3.0	.0000	3.959×10^{-6}	1.712×10^{-2}	.1687	.4175	.6377	.7919	.8890	.9463	.9783	.9967
3.5	.0000	4.543×10^{-6}	1.926×10^{-2}	.1857	.4496	.6734	.8226	.9115	.9605	.9857	.9984
4.0	.0000	5.115×10^{-6}	2.130×10^{-2}	.2013	.4781	.7036	.8472	.9283	.9705	.9904	.9992
4.5	.0000	5.677×10^{-6}	2.326×10^{-2}	.2159	.5036	.7295	.8673	.9414	.9776	.9934	.9996
5.0	.0000	6.231×10^{-6}	2.516×10^{-2}	.2295	.5266	.7520	.8839	.9516	.9828	.9954	.9998
6	.0000	7.317×10^{-6}	2.875×10^{-2}	.2545	.5669	.7891	.9096	.9662	.9896	.9977	.9999
7	.0000	8.377×10^{-6}	3.214×10^{-2}	.2769	.6010	.8185	.9284	.9758	.9935	.9988	1.0000
8	.0000	9.416×10^{-6}	3.536×10^{-2}	.2973	.6305	.8422	.9424	.9824	.9958	.9994	1.0000
9	.0000	1.044×10^{-5}	3.843×10^{-2}	.3160	.6563	.8618	.9532	.9871	.9973	.9997	1.0000
10	.0000	1.144×10^{-5}	4.137×10^{-2}	.3333	.6791	.8782	.9616	.9903	.9982	.9998	1.0000
11	.0000	1.242×10^{-5}	4.419×10^{-2}	.3439	.6994	.8920	.9683	.9926	.9988	.9999	1.0000
12	.0000	1.340×10^{-5}	4.691×10^{-2}	.3644	.7177	.9038	.9736	.9944	.9992	.9999	1.0000
13	.0000	1.436×10^{-5}	4.954×10^{-2}	.3785	.7343	.9140	.9779	.9957	.9995	1.0000	1.0000
14	.0000	1.530×10^{-5}	5.208×10^{-2}	.3918	.7493	.9228	.9814	.9967	.9996	1.0000	1.0000
15	.0000	1.624×10^{-5}	5.454×10^{-2}	.4043	.7630	.9306	.9844	.9975	.9998	1.0000	1.0000
16	.0000	1.716×10^{-5}	5.692×10^{-2}	.4161	.7756	.9373	.9867	.9980	.9998	1.0000	1.0000
17	.0000	1.808×10^{-5}	5.925×10^{-2}	.4274	.7872	.9432	.9887	.9985	.9999	1.0000	1.0000
18	.0000	1.899×10^{-5}	6.151×10^{-2}	.4381	.7978	.9485	.9904	.9988	.9999	1.0000	1.0000
19	.0000	1.988×10^{-5}	6.371×10^{-2}	.4482	.8077	.9532	.9918	.9990	.9999	1.0000	1.0000
20	.0000	2.077×10^{-5}	6.585×10^{-2}	.4580	.8169	.9574	.9930	.9992	1.0000	1.0000	1.0000

VALUE OF (ρ_a/ρ)

POINT SOURCE IN INFINITE MEDIUM

TABLE 19

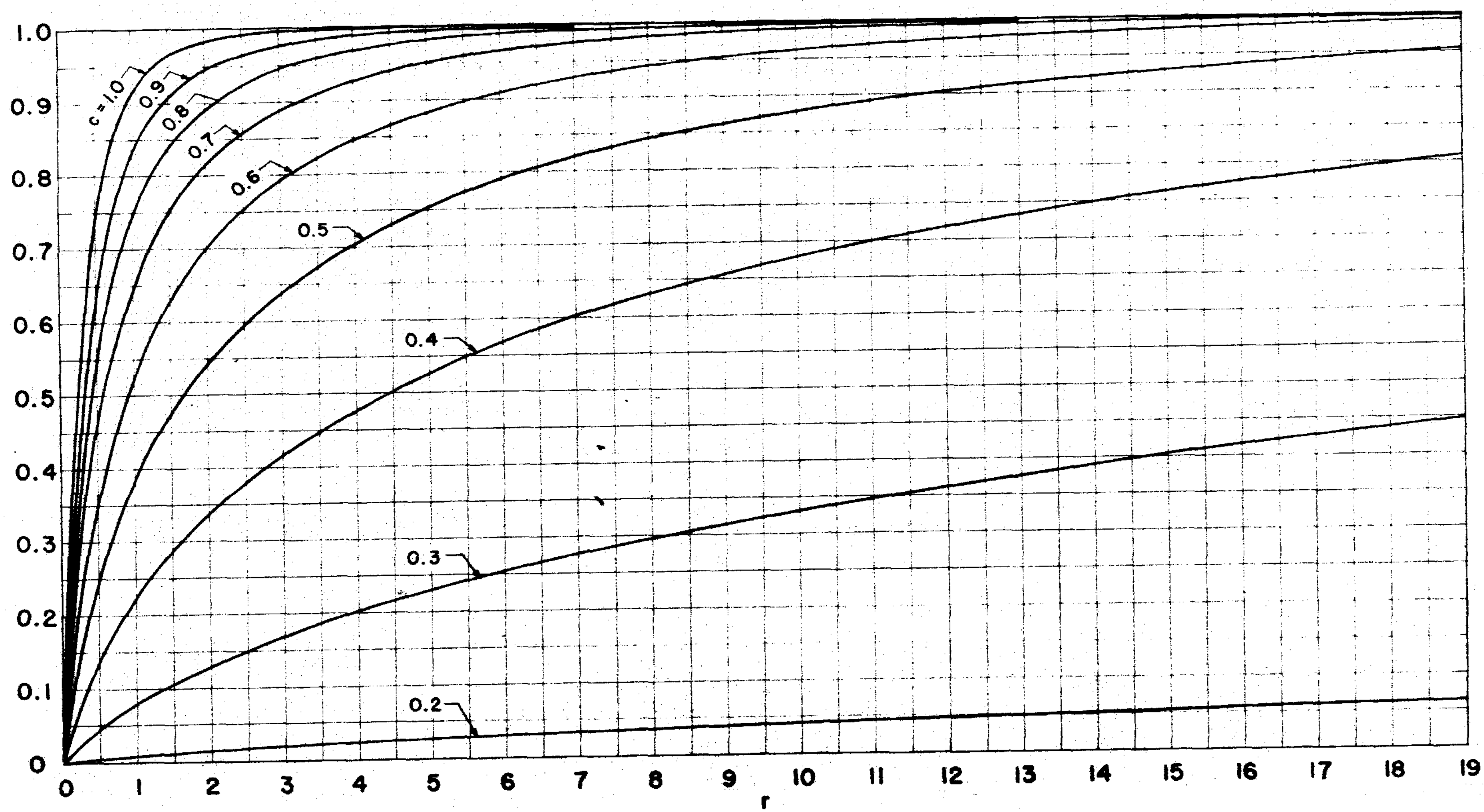


Fig. 28

ρ_{as}/ρ FOR POINT SOURCE IN INFINITE MEDIUM

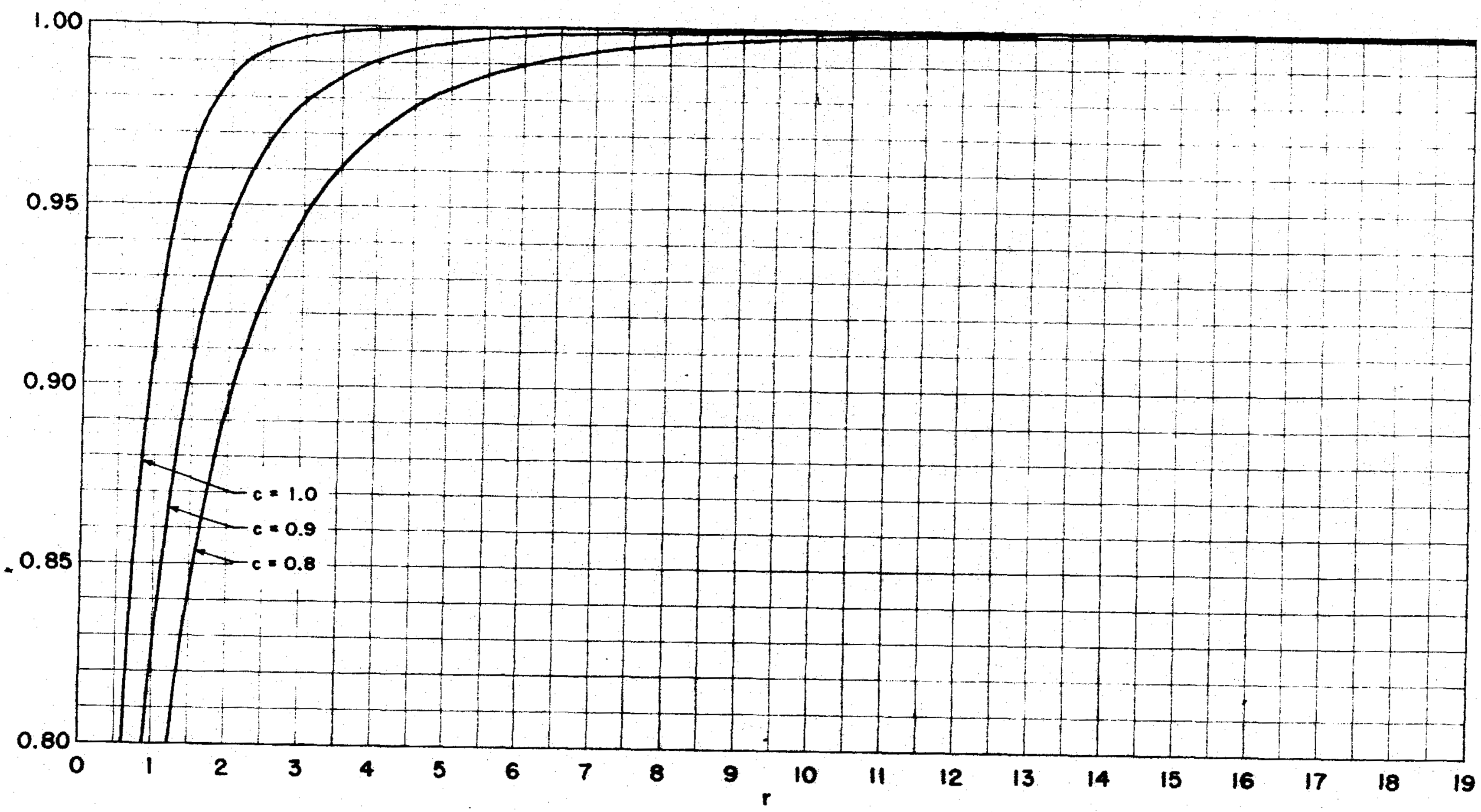


Fig. 29

ρ_{os}/ρ FOR POINT SOURCE IN INFINITE MEDIUM

For c = 0.5	at 1 m.f.p.	$p/\rho \sim 62\%$
	at 12 m.f.p.	$p/\rho \sim 10\%$
	at 20 m.f.p.	$p/\rho \sim 4.3\%$
For c = 0.3	at 1 m.f.p.	$p/\rho \sim 93\%$
	at 10 m.f.p.	$p/\rho \sim 65\%$
	at 20 m.f.p.	$p/\rho \sim 54\%$
For c = 0.1	Even at 20 m.f.p. the asymptotic density contributes only $2 \times 10^{-3}\%$.	

From these numbers it is seen that for large absorption (small c) the "asymptotic density" is a fiction. It only dominates at enormous distances where the total density is insignificant. Even for $c \sim 1$ one must go to several mean free paths before the non-asymptotic density becomes unimportant.

15. Isotropic Plane Source

To obtain the neutron density corresponding to an isotropic unit plane source at $z = 0$ we use the result of Appendix B. This tells us that the unit plane source solution with plane symmetry is related to that for the unit point source with spherical symmetry by:

$$\rho_{p\ell}(z) = 2\pi \int_{|z|}^{\infty} R\, \rho_p(R)\, dR \tag{1}$$

Using the particular solution (14-30) for the point source gives:

$$\begin{aligned} \rho_{p\ell}(z) &= -\frac{\partial K_o^2}{\partial c} \frac{e^{-K_o|z|}}{2K_o} + \frac{1}{2} \int_{|z|}^{\infty} dR \int_0^1 d\mu\, \frac{g(c,\mu)}{\mu^2}\, e^{-R/\mu} \\ &= -\frac{\partial K_o^2}{\partial c} \frac{e^{-K_o|z|}}{2K_o} + \frac{1}{2} \int_0^1 \frac{g(c,\mu)}{\mu}\, e^{-|z|/\mu}\, d\mu \end{aligned} \tag{2}$$

We see that the first term in (2) is dominant at large distances from the source and the second at small distances. Thus it is again convenient to decompose $\rho_{p\ell}(z)$ into an asymptotic part, $\rho_{as}(z)$, and a non-asymptotic part, $p(z)$, where

$$\rho_{p\ell}(z) = p(z) + \rho_{as}(z) \tag{3}$$

and

$$\rho_{as}(z) = -\frac{\partial K_o^2}{\partial c} \frac{e^{-K_o|z|}}{2K_o} \tag{4}$$

$$p(z) = \eta(z)\, \frac{1}{2}\, E_1(|z|) \tag{5}$$

with

$$\eta(z) = \frac{1}{E_1(|z|)} \int_0^1 \frac{g(c,\mu)}{\mu}\, e^{-|z|/\mu}\, d\mu \tag{6}$$

The relative contributions of the asymptotic and non-asymptotic densities are shown by Table 20 which gives values of $\rho_{as}(z)/\rho(z)$. Figures 30 and 31 are graphs of this ratio. The behavior is seen to be quite similar to the point source results (Table 19, Figures 28 and 29).

The general plane source solution with plane symmetry is obtained from (2) by adding the general plane symmetrical solution of the homogeneous equation. This we have seen is of the form $A e^{K_0 z} + B e^{-K_0 z}$ where A and B are arbitrary constants. Thus the general plane symmetrical solution is

$$\rho_{pl}(z) = -\frac{\partial K_0^2}{\partial c}\frac{e^{-K_0|z|}}{2K_0} + \frac{1}{2}\int_0^1 \frac{g(c,\mu)}{\mu} e^{-|z|/\mu} d\mu + A e^{K_0 z} + B e^{-K_0 z} \qquad (7)$$

The asymptotic solution is then

$$\rho_{as}(z) = \begin{cases} \left(B - \frac{1}{2K_0}\frac{\partial K_0^2}{\partial c}\right) e^{-K_0 z} + A e^{K_0 z} & z > 0 \\ B e^{-K_0 z} + \left(A - \frac{1}{2K_0}\frac{\partial K_0^2}{\partial c}\right) e^{K_0 z} & z < 0 \end{cases} \qquad (8)$$

The solution for large positive z is thus obtained from that for large negative z by adding $\frac{1}{K_0}\frac{\partial K_0^2}{\partial c} \sinh K_0 z$. However, $\sinh K_0 z$ is just an infinite medium solution. In other words, the effect of the plane source is to cause a discontinuity in the infinite medium solutions which hold far from the source.

The general plane source solution irrespective of symmetry is obtained by adding the general solution (13-7) of the homogeneous equation (13-1). Hence the general plane source solution is:

$$\rho_{pl}(\vec{r}) = -\frac{\partial K_0^2}{\partial c}\frac{e^{-K_0|z|}}{2K_0} + \frac{1}{2}\int_0^1 \frac{g(c,\mu)e^{-|z|/\mu}}{\mu} d\mu + \int f(\vec{u}) e^{-K_0 \vec{u}\cdot\vec{r}} du \qquad (9)$$

x \ e	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	1.0000
.1	.0000	7.654×10^{-7}	3.926×10^{-3}	4.639×10^{-2}	.1380	.2519	.3703	.4874	.6046	.7321	1.0000
.2	.0000	1.014×10^{-6}	5.095×10^{-3}	5.599×10^{-2}	.1717	.3066	.4402	.5653	.6825	.7989	1.0000
.3	.0000	1.221×10^{-6}	6.046×10^{-3}	6.888×10^{-2}	.1972	.3463	.4891	.6175	.7320	.8387	1.0000
.4	.0000	1.408×10^{-6}	6.883×10^{-3}	7.738×10^{-2}	.2185	.3785	.5274	.6570	.7680	.8663	1.0000
.5	.0000	1.582×10^{-6}	7.650×10^{-3}	8.498×10^{-2}	.2372	.4059	.5592	.6887	.7960	.8867	1.0000
.6	.0000	1.748×10^{-6}	8.365×10^{-3}	9.196×10^{-2}	.2539	.4300	.5863	.7151	.8186	.9030	1.0000
.7	.0000	1.907×10^{-6}	9.042×10^{-3}	9.845×10^{-2}	.2691	.4514	.6100	.7376	.8373	.9159	1.0000
.8	.0000	2.060×10^{-6}	9.688×10^{-3}	.1045	.2832	.4709	.6310	.7571	.8532	.9265	1.0000
.9	.0000	2.209×10^{-6}	1.031×10^{-2}	.1103	.2963	.4886	.6499	.7742	.8668	.9353	1.0000
1.0	.0000	2.355×10^{-6}	1.092×10^{-2}	.1158	.3085	.5050	.6669	.7894	.8786	.9427	1.0000
1.5	.0000	3.045×10^{-6}	1.367×10^{-2}	.1402	.3609	.5719	.7334	.8457	.9199	.9669	1.0000
2.0	.0000	3.693×10^{-6}	1.615×10^{-2}	.1611	.4031	.6223	.7799	.8823	.9442	.9796	1.0000
2.5	.0000	4.313×10^{-6}	1.845×10^{-2}	.1795	.4387	.6625	.8148	.9075	.9598	.9869	1.0000
3.0	.0000	4.913×10^{-6}	2.062×10^{-2}	.1963	.4695	.6955	.8418	.9260	.9703	.9913	1.0000
3.5	.0000	5.497×10^{-6}	2.266×10^{-2}	.2117	.4967	.7233	.8635	.9400	.9777	.9942	1.0000
4.0	.0000	6.069×10^{-6}	2.463×10^{-2}	.2260	.5210	.7472	.8812	.9507	.9830	.9960	1.0000
4.5	.0000	6.630×10^{-6}	2.651×10^{-2}	.2392	.5423	.7679	.8958	.9592	.9869	.9972	1.0000
5.0	.0000	7.181×10^{-6}	2.833×10^{-2}	.2517	.5629	.7860	.9082	.9659	.9898	.9980	1.0000
6	.0000	8.260×10^{-6}	3.179×10^{-2}	.2747	.5981	.8164	.9275	.9758	.9937	.9990	1.0000
7	.0000	9.314×10^{-6}	3.506×10^{-2}	.2955	.6283	.8408	.9420	.9824	.9960	.9995	1.0000
8	.0000	1.034×10^{-5}	3.817×10^{-2}	.3145	.6545	.8608	.9529	.9871	.9974	.9997	1.0000
9	.0000	1.136×10^{-5}	4.114×10^{-2}	.3321	.6777	.8774	.9615	.9904	.9983	.9999	1.0000
10	.0000	1.235×10^{-5}	4.399×10^{-2}	.3483	.6983	.8915	.9682	.9928	.9989	.9999	1.0000
11	.0000	1.333×10^{-5}	4.673×10^{-2}	.3635	.7168	.9034	.9736	.9945	.9992	1.0000	1.0000
12	.0000	1.429×10^{-5}	4.936×10^{-2}	.3777	.7335	.9137	.9779	.9958	.9995	1.0000	1.0000
13	.0000	1.524×10^{-5}	5.191×10^{-2}	.3910	.7486	.9226	.9814	.9968	.9997	1.0000	1.0000
14	.0000	1.619×10^{-5}	5.438×10^{-2}	.4036	.7624	.9304	.9843	.9975	.9998	1.0000	1.0000
15	.0000	1.711×10^{-5}	5.677×10^{-2}	.4155	.7751	.9372	.9867	.9981	.9998	1.0000	1.0000
16	.0000	1.803×10^{-5}	5.908×10^{-2}	.4268	.7867	.9431	.9887	.9985	.9999	1.0000	1.0000
17	.0000	1.894×10^{-5}	6.137×10^{-2}	.4375	.7974	.9484	.9904	.9988	.9999	1.0000	1.0000
18	.0000	1.984×10^{-5}	6.357×10^{-2}	.4477	.8074	.9531	.9918	.9991	1.0000	1.0000	1.0000
19	.0000	2.072×10^{-5}	6.572×10^{-2}	.4575	.8166	.9573	.9929	.9993	1.0000	1.0000	1.0000
20	.0000	2.160×10^{-5}	6.781×10^{-2}	.4668	.8251	.9611	.9939	.9994	1.0000	1.0000	1.0000

VALUE OF (ρ_s/ρ)

PLANE SOURCE IN INFINITE MEDIUM

TABLE 20

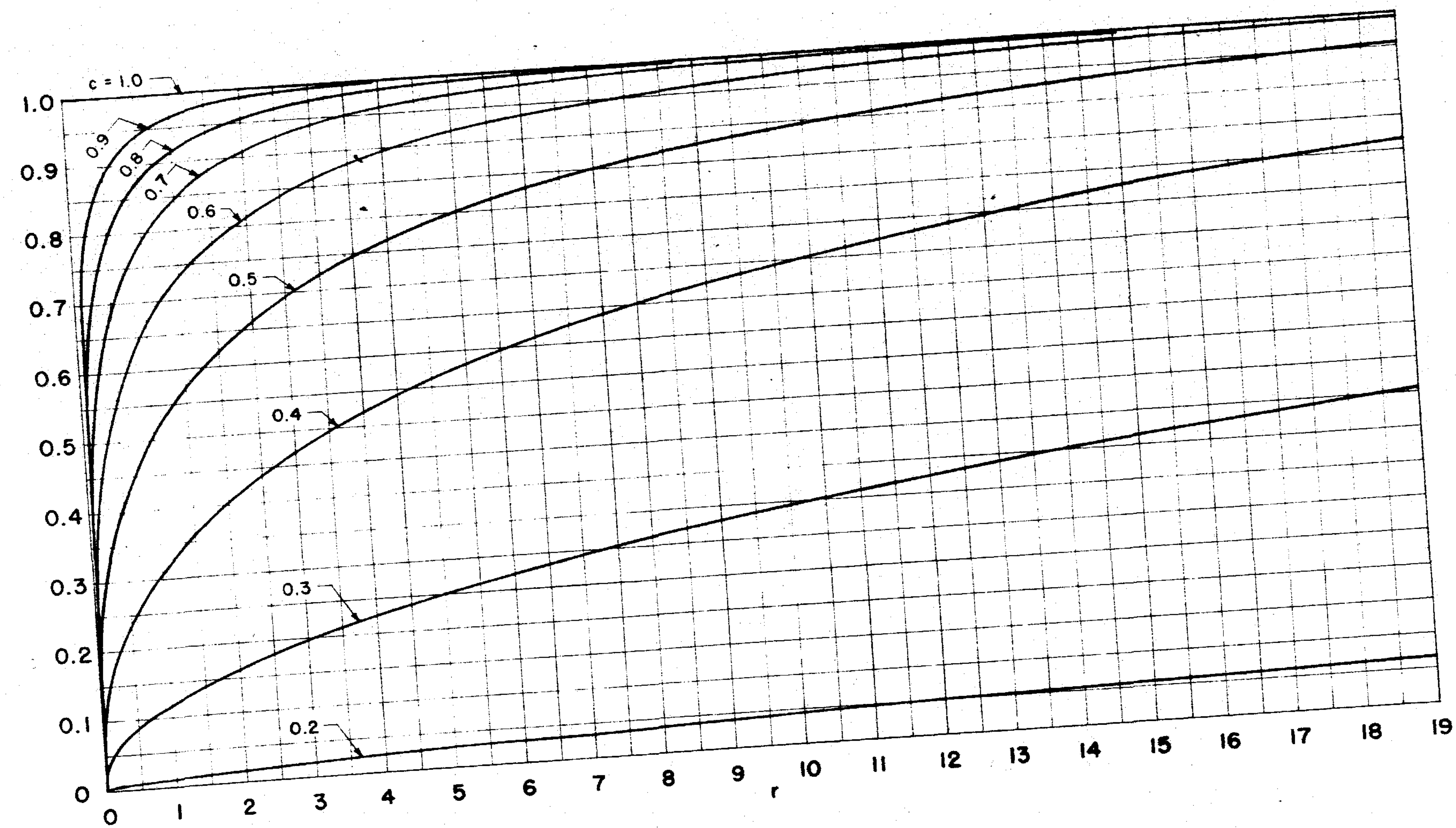


Fig. 30
ρ_{as}/ρ FOR PLANE SOURCE IN INFINITE MEDIUM

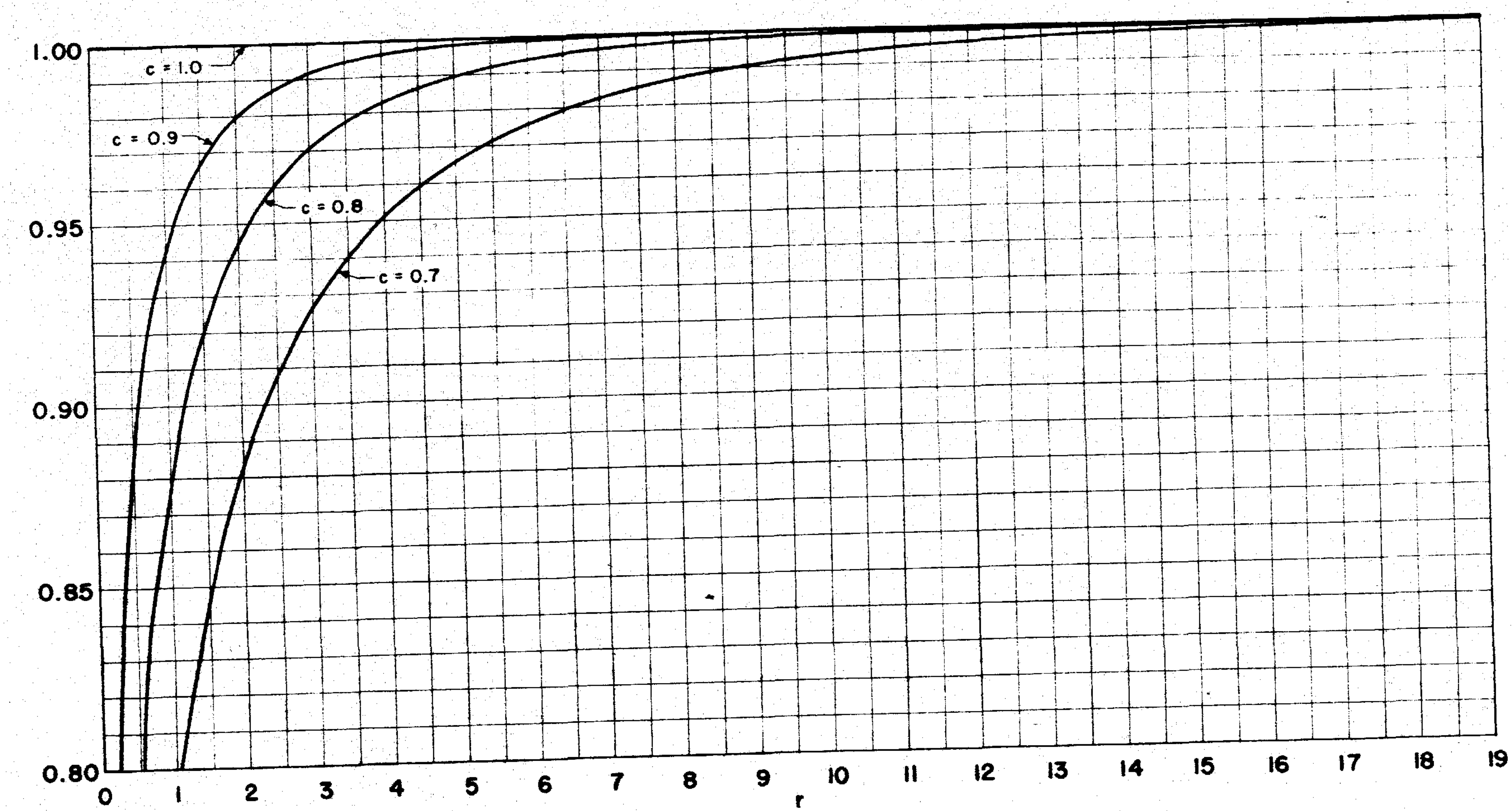


Fig. 31

ρ_{as}/ρ FOR PLANE SOURCE IN INFINITE MEDIUM where $f(\vec{u})$ is arbitrary.

Returning to the particular solution (z) we note that the various even moments for the plane case are readily obtainable since, as shown in Appendix B, they are related to those for the point source by

$$\int_{-\infty}^{\infty} z^{2n} \rho_{p\ell}(z)\, dz = \frac{1}{2n+1} \int \rho_p(R)\, d\vec{R} \tag{10}$$

The 2n'th moment of one of the plane source densities is just $\frac{1}{2n+1}$ times that of the corresponding point source density.

Proceeding as in section 14 we find that for small distances the solution (2) is approximately

$$\rho_{p\ell}(z) \approx -\frac{\log|z|}{2} + \frac{1}{2}\left\{-\gamma - \frac{1}{K_o}\frac{\partial K_o^2}{\partial c} + I(c)\right\} \tag{11}$$

$$+ \frac{|z|}{2}\left\{1 - \frac{c\pi^2}{4}\right\} + 0\left(|z|^2 \log |z|\right) \qquad (z \ll 1)$$

where $\gamma = 0.577216$

$$\text{and} \quad I(c) = \int_0^1 \left\{g(c,\mu) - 1\right\} \frac{d\mu}{\mu} \tag{12}$$

The function I(c) is shown in Figure 32.

The non-asymptotic density may be put in a convenient form by inserting the previously obtained expansion for g

$$g(c,\mu) = 1 + \sum_{n=1}^{\infty} \Gamma_n(c)\mu^{2n} \tag{14-58}$$

(where $\Gamma_n(c)$ are given in Table 14), into the definition of $\rho(z)$. This gives:

$$\rho(z) = \frac{1}{2}\left\{E_1(|z|) + \sum_{n=1}^{\infty} \Gamma_n(c)\, E_{2n+1}(|z|)\right\} \tag{13}$$

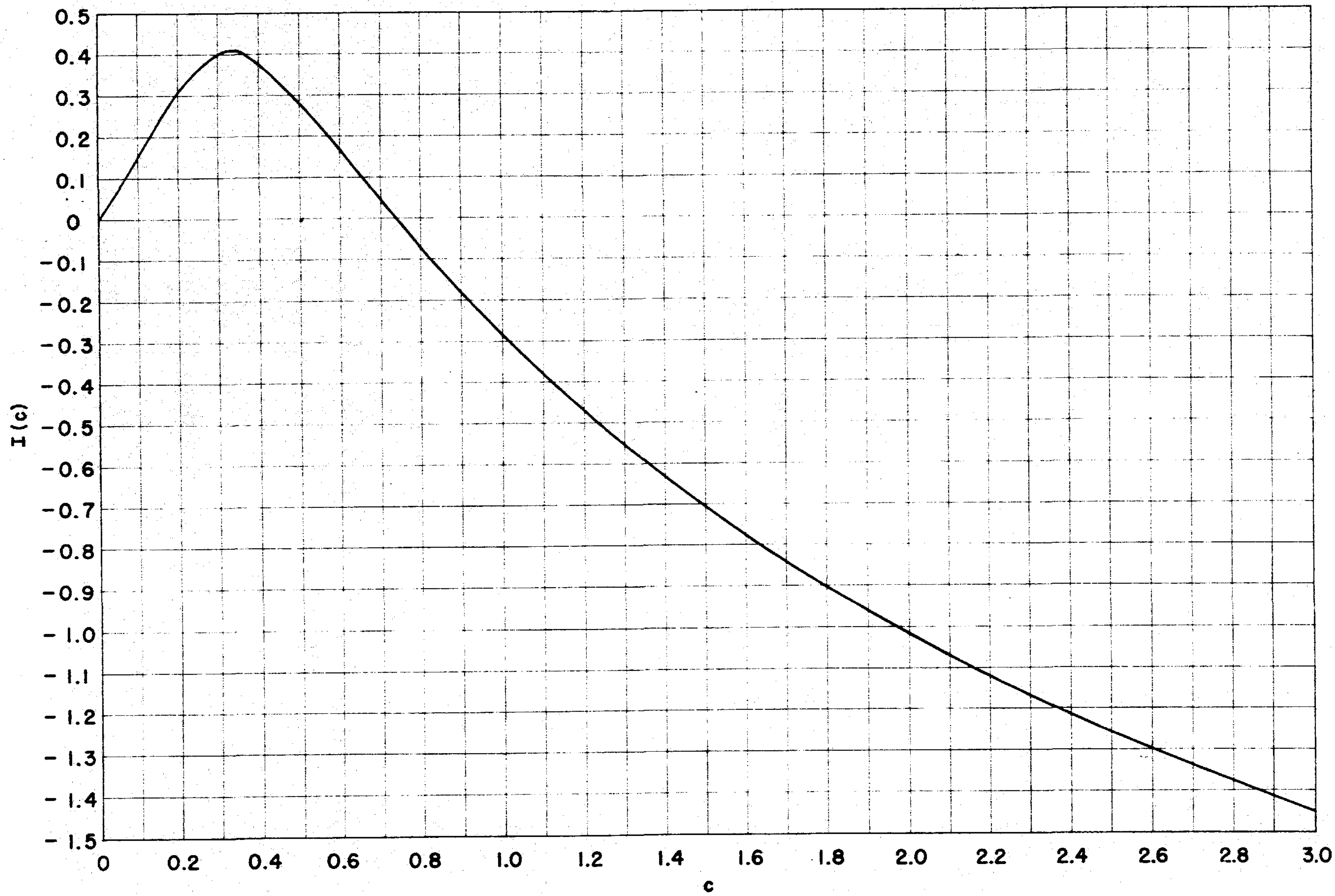

0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
-1.1
-1.2
-1.3
-1.4
-1.5
I (c)
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
c

Fig. 32

The detailed features of the non-asymptotic density are best shown in the function $\eta(z)$. From the expansion (11) we find that for small z

$$\eta(z) \approx \frac{1}{-\log z - \gamma + z} \left\{ -\log z + \left[-\gamma + I(c)\right] + z\left[1 - \frac{c\pi^2}{4} - \frac{\partial K_o^2}{\partial c}\right]\right\}$$

$$\approx 1 - \frac{I(c)}{\log z} \qquad (z \ll 1) \qquad (14)$$

Hence

$$\eta(o) = 1 \qquad (15)$$

and

$$\eta'(z) \approx \frac{I(c)}{z(\log z)^2} \qquad (z \ll 1) \qquad (16)$$

From Figure 32 we see I(c) is zero at c=0 (where $\eta(z) \equiv 1$), rises to a positive maximum of 0.412 at c=0.330, vanishes at c = 0.731, and is thenceforth negative (being asymptotically proportional to $-c$ for large c). For c = 0, $\eta'(o)$ is zero. For c between zero and 0.731, $\eta'(o)$ is $+\infty$. For small z the curves of $\eta(z)$ will be successively higher with increasing c until c = 0.330. Then they will lie lower with increasing c until c = 0.731 where the initial slope is zero. For $c > 0.731$ the initial slope is $-\infty$ and the curves of $\eta(z)$ with increasing c will lie successively lower. This behavior is shown in Figure 33.

When z is large we find on inserting the asymptotic expansion of the E_1 function in (5):

$$\eta(z) \approx z\, e^z \int_0^1 \frac{g(c,\mu)}{\mu} e^{-z/\mu}\, d\mu \qquad (z \gg 1) \qquad (17)$$

Introducing $x = \frac{1}{\mu} - 1$ as integration variable and noting that only the region $x \ll 1$ will give an appreciable contribution for large z enables us to write (to the same approximation as that leading to (14-78))

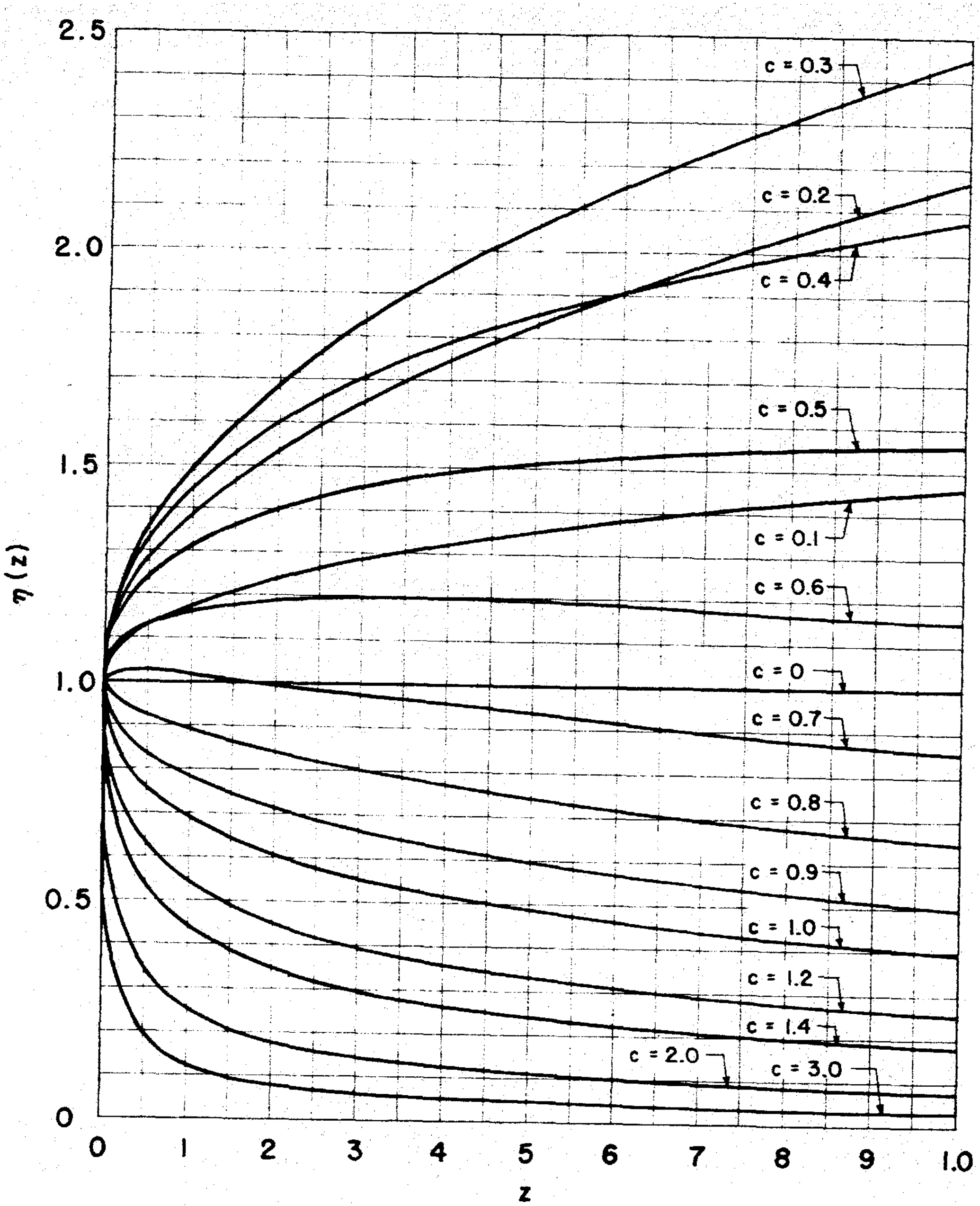


Fig. 33
PLANE SOURCE IN INFINITE MEDIUM

$$\eta(z) \approx z \int_0^\infty \bar{g}(c,x)\, e^{-zx}\, dx \qquad (z \gg 1) \qquad (18)$$

where $\bar{g}$ is given by (14-75). Thus, on comparing with (14-78) it is seen that asymptotically $\eta(z)$ behaves just as $\epsilon(r)$ and the discussion of section 14 may be taken over completely.

In the region where z is neither large nor small, $\eta(z)$ must be obtained by numerical integration. Figure 33 shows the resulting functions. Values of the function are given in Table 21.

0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	1.0000	1.9694	1.1454	1.1890	1.1760	1.1315	1.0760	1.0192	.9648	.9143	.8679
.2	1.0000	1.0894	1.1892	1.2451	1.2271	1.1676	1.0956	1.0223	.9531	.8896	.8320
.3	1.0000	1.1048	1.2232	1.2892	1.2666	1.1956	1.1096	1.0237	.9434	.8706	.8051
.4	1.0000	1.1179	1.2524	1.3271	1.3001	1.2185	1.1208	1.0241	.9348	.8545	.7830
.5	0.0000	1.1294	1.2785	1.3608	1.3295	1.2382	1.1300	1.0240	.9269	.8425	.7640
.6	1.0000	1.1392	1.3022	1.3916	1.3561	1.2557	1.1378	1.0234	.9196	.8277	.7472
.7	1.0000	1.1494	1.3243	1.4200	1.3804	1.2713	1.1446	1.0226	.9126	.8161	.7322
.8	1.0000	1.1582	1.3449	1.4467	1.4030	1.2856	1.1505	1.0215	.9060	.8054	.7185
.9	1.0000	1.1665	1.3643	1.4718	1.4241	1.2987	1.1558	1.0202	.8997	.7954	.7059
1.0	1.0000	1.1743	1.3813	1.4956	1.4439	1.3108	1.1604	1.0187	.8936	.7860	.6942
1.5	1.0000	1.2081	1.4643	1.6003	1.5286	1.3605	1.1772	1.0099	.8664	.7462	.6462
2.0	1.0000	1.2358	1.5334	1.6883	1.5971	1.3977	1.1871	.9997	.8428	.7142	.6094
2.5	1.0000	1.2596	1.5943	1.7653	1.6548	1.4269	1.1927	.9890	.8219	.6874	.5797
3.0	1.0000	1.2806	1.6492	1.8343	1.7046	1.4505	1.1955	.9782	.8030	.6644	.5548
3.5	1.0000	1.2995	1.6996	1.8970	1.7484	1.4697	1.1963	.9675	.7859	.6441	.5335
4.0	1.0000	1.3167	1.7464	1.9547	1.7874	1.4857	1.1957	.9569	.7701	.6261	.5150
4.5	1.0000	1.3326	1.7901	2.0084	1.8225	1.4989	1.1939	.9465	.7555	.6099	.4986
5.0	1.0000	1.3473	1.8314	2.0585	1.8542	1.5100	1.1913	.9364	.7419	.5952	.4840
6.0	1.0000	1.3741	1.9079	2.1502	1.9097	1.5271	1.1843	.9169	.7173	.5693	.4588
7.0	1.0000	1.3980	1.9779	2.2329	1.9568	1.5390	1.1756	.8986	.6954	.5470	.4378
8.0	1.0000	1.4197	2.0427	2.3083	1.9972	1.5469	1.1658	.8812	.6757	.5277	.4198
9.0	1.0000	1.4397	2.1034	2.3779	2.0324	1.5520	1.1554	.8647	.6579	.5106	.4042
10.0	1.0000	1.4582	2.1607	2.4427	2.0632	1.5546	1.1446	.8491	.6416	.4952	.3904
11.0	1.0000	1.4755	2.2152	2.5035	2.0905	1.5554	1.1335	.8342	.6265	.4813	.3781
12.0	1.0000	1.4919	2.2673	2.5608	2.1149	1.5546	1.1222	.8200	.6125	.4687	.3670
13.0	1.0000	1.5074	2.3172	2.6151	2.1366	1.5527	1.1109	.8065	.5995	.4570	.3570
14.0	1.0000	1.5221	2.3653	2.6669	2.1563	1.5497	1.0995	.7934	.5872	.4463	.3477
15.0	1.0000	1.5363	2.4119	2.7164	2.7741	1.5459	1.0882	.7809	.5757	.4362	.3392
16.0	1.0000	1.5498	2.4570	2.7639	2.1902	1.5413	1.0769	.7689	.5647	.4268	.3313
17.0	1.0000	1.5629	2.5006	2.8094	2.2049	1.5362	1.0657	.7572	.5544	.4181	.3239
18.0	1.0000	1.5754	2.5431	2.8534	2.2183	1.5306	1.0546	.7460	.5445	.4098	.3171
19.0	1.0000	1.5876	2.5845	2.8959	2.2306	1.5246	1.0436	.7351	.5351	.4019	.3106
20.0	1.0000	1.5994	2.6248	2.9370	2.2420	1.5182	1.0327	.7246	.5261	.3945	.3045

$\eta(z)$ for Plane Source

(Sheet 1)

TABLE 21

z \ c	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	.8679	.8253	.7863	.7506	.7177	.6874	.6593	.6333	.6091	.5866	.5655
.2	.8320	.7798	.7325	.6898	.6509	.6154	.5830	.5533	.5259	.5007	.4774
.3	.8051	.7465	.6940	.6469	.6045	.5662	.5315	.5000	.4712	.4450	.4209
.4	.7830	.7196	.6633	.6132	.5685	.5285	.4925	.4601	.4308	.4041	.3799
.5	.7640	.6968	.6377	.5854	.5392	.4980	.4613	.4285	.3989	.3723	.3481
.6	.7472	.6770	.6156	.5618	.5144	.4726	.4355	.4024	.3729	.3464	.3226
.7	.7322	.6594	.5962	.5412	.4931	.4508	.4135	.3805	.3511	.3249	.3014
.8	.7185	.6436	.5790	.5230	.4744	.4318	.3945	.3616	.3325	.3067	.2836
.9	.7059	.6292	.5634	.5068	.4578	.4151	.3779	.3452	.3164	.2909	.2683
1.0	.6942	.6161	.5493	.4921	.4429	.4002	.3632	.3308	.3023	.2772	.2550
1.5	.6462	.5631	.4936	.4354	.3861	.3443	.3086	.2778	.2512	.2281	.2078
2.0	.6094	.5239	.4537	.3957	.3474	.3069	.2727	.2436	.2187	.1973	.1787
2.5	.5797	.4931	.4229	.3657	.3187	.2796	.2469	.2194	.1960	.1760	.1588
3.0	.5548	.4679	.3983	.3421	.2963	.2587	.2274	.2012	.1791	.1603	.1442
3.5	.5335	.4467	.3780	.3229	.2784	.2420	.2119	.1869	.1659	.1481	.1329
4.0	.5150	.4286	.3608	.3068	.2635	.2283	.1994	.1754	.1553	.1384	.1240
4.5	.4986	.4129	.3460	.2931	.2509	.2168	.1889	.1658	.1466	.1304	.1167
5.0	.4840	.3990	.3331	.2813	.2401	.2070	.1800	.1577	.1393	.1237	.1106
6.0	.4588	.3754	.3115	.2617	.2225	.1911	.1656	.1448	.1275	.1131	.1009
7.0	.4378	.3561	.2940	.2461	.2085	.1786	.1545	.1348	.1185	.1049	.0935
8.0	.4198	.3400	.2796	.2333	.1972	.1685	.1455	.1268	.1114	.0985	.0878
9.0	.4042	.3259	.2673	.2225	.1876	.1601	.1381	.1202	.1054	.0932	.0830
10.0	.3904	.3138	.2567	.2132	.1796	.1530	.1318	.1146	.1005	.0888	.0790
11.0	.3781	.3031	.2474	.2052	.1725	.1469	.1264	.1099	.0963	.0851	.0757
12.0	.3670	.2935	.2391	.1980	.1665	.1415	.1217	.1058	.0926	.0818	.0727
13.0	.3570	.2849	.2318	.1917	.1609	.1368	.1176	.1021	.0894	.0790	.0702
14.0	.3477	.2770	.2251	.1860	.1560	.1325	.1139	.0989	.0866	.0764	.0679
15.0	.3392	.2698	.2189	.1808	.1515	.1287	.1105	.0959	.0840	.0741	.0659
16.0	.3313	.2632	.2133	.1760	.1475	.1252	.1075	.0933	.0816	.0721	.0640
17.0	.3239	.2570	.2082	.1717	.1438	.1220	.1048	.0909	.0796	.0702	.0624
18.0	.3171	.2513	.2034	.1677	.1404	.1191	.1023	.0887	.0776	.0685	.0609
19.0	.3106	.2460	.1990	.1639	.1372	.1164	.0999	.0867	.0759	.0670	.0595
20.0	.3045	.2409	.1948	.1605	.1343	.1139	.0978	.0849	.0742	.0655	.0583

η (z) For Plane Source

(Sheet 2)

TABLE 21

z \ c	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	.5655	.5458	.5272	.5098	.4934	.4778	.4631	.4493	.4361	.4236	.4117
.2	.4774	.4557	.4357	.4169	.3995	.3832	.3679	.3535	.3400	.3273	.3154
.3	.4209	.3987	.3783	.3595	.3420	.3258	.3107	.2967	.2836	.2713	.2508
.4	.3799	.3577	.3374	.3188	.3017	.2859	.2713	.2578	.2452	.2336	.2227
.5	.3481	.3262	.3063	.2881	.2714	.2562	.2421	.2291	.2172	.2061	.1958
.6	.3226	.3010	.2815	.2638	.2477	.2329	.2194	.2070	.1956	.1851	.1754
.7	.3014	.2803	.2613	.2441	.2284	.2142	.2012	.1893	.1785	.1685	.1592
.8	.2836	.2629	.2444	.2276	.2125	.1988	.1863	.1749	.1645	.1549	.1462
.9	.2683	.2481	.2300	.2137	.1990	.1858	.1738	.1628	.1529	.1437	.1354
1.0	.2550	.2352	.2176	.2018	.1875	.1746	.1631	.1526	.1430	.1343	.1263
1.5	.2078	.1901	.1744	.1605	.1481	.1371	.1272	.1183	.1103	.1030	.0964
2.0	.1787	.1626	.1484	.1360	.1250	.1152	.1065	.0987	.0918	.0855	.0798
2.5	.1588	.1439	.1309	.1196	.1096	.1008	.0930	.0860	.0797	.0741	.0691
3.0	.1442	.1303	.1183	.1078	.0986	.0905	.0833	.0770	.0713	.0662	.0616
3.5	.1329	.1199	.1087	.0989	.0903	.0828	.0762	.0703	.0650	.0603	.0561
4.0	.1240	.1117	.1011	.0918	.0838	.0768	.0705	.0650	.0601	.0558	.0518
4.5	.1167	.1050	.0949	.0862	.0786	.0719	.0660	.0608	.0562	.0521	.0484
5.0	.1106	.0994	.0898	.0814	.0742	.0679	.0623	.0574	.0530	.0491	.0456
6.0	.1009	.0906	.0817	.0740	.0674	.0616	.0565	.0520	.0480	.0444	.0413
7.0	.0935	.0839	.0756	.0684	.0623	.0569	.0521	.0480	.0443	.0410	.0380
8.0	.0878	.0786	.0708	.0641	.0583	.0532	.0488	.0449	.0414	.0383	.0356
9.0	.0830	.0743	.0669	.0605	.0550	.0502	.0460	.0423	.0390	.0361	.0335
10.0	.0790	.0707	.0637	.0576	.0524	.0478	.0438	.0403	.0371	.0344	.0319
11.0	.0757	.0677	.0609	.0551	.0501	.0457	.0419	.0385	.0355	.0328	.0305
12.0	.0727	.0651	.0586	.0530	.0481	.0439	.0402	.0370	.0341	.0316	.0293
13.0	.0702	.0628	.0565	.0511	.0464	.0424	.0388	.0357	.0329	.0305	.0283
14.0	.0679	.0607	.0546	.0494	.0449	.0410	.0375	.0345	.0318	.0295	.0273
15.0	.0659	.0589	.0530	.0479	.0436	.0397	.0364	.0335	.0309	.0286	.0265
16.0	.0640	.0573	.0515	.0466	.0424	.0386	.0354	.0326	.0300	.0278	.0258
17.0	.0624	.0558	.0502	.0454	.0413	.0377	.0345	.0318	.0293	.0271	.0252
18.0	.0609	.0545	.0490	.0443	.0403	.0368	.0337	.0310	.0286	.0265	.0246
19.0	.0595	.0533	.0479	.0434	.0394	.0360	.0330	.0304	.0280	.0260	.0241
20.0	.0583	.0521	.0469	.0425	.0386	.0353	.0323	.0298	.0275	.0255	.0237

$\eta(z)$ For Plane Source

(Sheet 3)

TABLE 21

16. Anisotropic Sources

.1 Anisotropic Plane Source

Consider a unit anisotropic plane source

$$q(\vec{r},\vec{\Omega}) = \delta(z)\, f(\vec{\Omega}) \qquad (1)$$

where

$$\int f(\vec{\Omega})\, d\Omega = 1 \qquad (2)$$

As remarked in section 12, this problem may be solved by dividing the neutron density into two parts

$$\rho(z) = \rho_d(z) + \rho_{id}(z) \qquad (3)$$

where $\rho_d(z)$ is the density of neutrons which have come from the source without having suffered a collision and ρ_{id} is the remainder.

Inserting (1) into (5-6) using (4-8) gives for the angular density of neutrons coming from the source without a collision

$$\psi(z,\vec{\Omega}) = f(\vec{\Omega})\, \frac{e^{-|z|/\nu}}{\nu} \qquad \nu > 0 \qquad (4)$$

$$= 0 \qquad \nu < 0$$

with $\nu = \dfrac{\vec{\Omega}\cdot\vec{z}}{|z|}$

On integrating over $\vec{\Omega}$ the "direct" contribution to the neutron density is:

$$\rho_d(z) = \int_{\nu>0} f(\vec{\Omega})\, \frac{e^{-|z|/\nu}}{\nu}\, d\Omega \qquad (5)$$

These neutrons on being scattered act as an isotropic distributed plane source of strength

$$q_{eff}(z) = c\, \rho_d(z) \qquad (6)$$

Hence the total neutron density due to a unit anisotropic plane source is:

$$\rho_{p\ell}^{\text{aniso.}}(z) = \rho_d(z) + c \int_{-\infty}^{\infty} \rho_{p\ell}(|z - z'|) \; \rho_d(z')\, dz' \tag{7}$$

Here $\rho_{p\ell}(z)$ is the isotropic plane source solution previously obtained (15-2). When $f(\vec{\Omega})$ is non-singular the asymptotic and non-asymptotic portions of the solution may be obtained by inserting the separate terms of (15-3).

Thus:

$$\left(\begin{matrix}\text{aniso.}\\ \rho_{p\ell}\end{matrix}\right)_{n-as} = \rho_d(z) + c \int_{-\infty}^{\infty} p\,(|z - z'|) \; \rho_d(z')\, dz' \tag{8}$$

while

$$\left(\begin{matrix}\text{aniso.}\\ \rho_{p\ell}\end{matrix}\right)_{as} = c \int_{-\infty}^{\infty} \rho_{as}(|z - z'|) \; \rho_d(z')\, dz' \tag{9}$$

Inserting ρ_{as} gives for the asymptotic solution

$$\left(\begin{matrix}\text{aniso.}\\ \rho_{p\ell}\end{matrix}\right)_{as} = - \frac{\partial K_o^2}{\partial c} \frac{e^{-K_o|z|}}{2 K_o} J \tag{10}$$

where

$$J = c \int_{-\infty}^{\infty} e^{-K_o(|z - z'| - |z|)}\, dz' \int_{\nu>0} f(\vec{\Omega}) \frac{e^{-|z'|/\nu}}{\nu} \, d\Omega \tag{11}$$

Hence the asymptotic solution is similar in form to that for the isotropic source. However, the constant multiplying $e^{-K_o|z|}$ may be different for z large and positive from what it is for z large and negative.

If we introduce coordinates for $\vec{\Omega}$ as in section 10.2 and let

$$\bar{f}(\mu) = \int_0^{2\pi} f(\vec{\Omega})\, d\phi \tag{12}$$

equation (11) becomes

$$J = c \int_{-\infty}^{\infty} e^{-K_0(|z - z'| - |z|)} dz' \int_{\mu z' > 0} \frac{\bar{f}(\mu)\, e^{-z'/\mu}}{|\mu|} d\mu \tag{13}$$

For $z \rightarrow +\infty$ $\quad |z - z'| - |z| \rightarrow -z'$ (14)

Decomposing the z' integral into portions from positive and negative regions gives:

$$J = c\left\{ \int_0^{\infty} e^{K_0 z'} dz' \int_0^1 \bar{f}(\mu)\, e^{-z'/\mu} \frac{d\mu}{\mu} - \int_{-\infty}^{0} e^{K_0 z'} dz' \int_{-1}^{0} \bar{f}(\mu) e^{-z'/\mu} \frac{d\mu}{\mu} \right\} \tag{15}$$

Carrying out the z' integration gives

$$\begin{aligned} J &= c\left\{ \int_0^1 \frac{\bar{f}(\mu)}{\mu} \frac{1}{\frac{1}{\mu} - K_0} d\mu - \int_{-1}^{0} \frac{\bar{f}(\mu)}{\mu} \frac{1}{K_0 - \frac{1}{\mu}} d\mu \right\} \\ &= c \int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 - K_0 \mu} \end{aligned} \tag{16}$$

Similarly for $z < 0$ we find

$$J = c \int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 + K_0 \mu} = c \int_{-1}^{1} \frac{\bar{f}(-\mu)\, d\mu}{1 - K_0 \mu} \tag{17}$$

The general plane symmetric solution corresponding to an anisotropic plane source is obtained by adding to (8) a solution of the homogeneous equation

$$\rho_h = A \cosh K_0 z + B \sinh K_0 z \tag{18}$$

where A and B are arbitrary.

For z large and positive the general solution is then

$$\rho \sim A \cosh K_o z + B \sinh K_o z - \left[c \int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 - K_o \mu} \right] \frac{\partial K_o^2}{\partial c} \frac{e^{-K_o z}}{2 K_o}$$

$$= \left\{ A - \frac{c}{2K_o} \frac{\partial K_o^2}{\partial c} \left[\int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 - K_o \mu} \right] \right\} \cosh K_o z$$

$$+ \left\{ B + \frac{c}{2K_o} \frac{\partial K_o^2}{\partial c} \left[\int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 - K_o \mu} \right] \right\} \sinh K_o z \qquad (19)$$

For z large and negative

$$\rho \sim \left\{ A - \frac{c}{2K_o} \frac{\partial K_o^2}{\partial c} \left[\int_{-1}^{1} \frac{\bar{f}(-\mu)\, d\mu}{1 - K_o \mu} \right] \right\} \cosh K_o z$$

$$+ \left\{ B - \frac{c}{2K_o} \frac{\partial K_o^2}{\partial c} \left[\int_{-1}^{1} \frac{\bar{f}(-\mu)\, d\mu}{1 - K_o \mu} \right] \right\} \sinh K_o z \qquad (20)$$

The solution for large positive z is thus obtained from that for large negative z by adding ρ_h'

$$\rho_h' = -\frac{c}{K_o} \frac{\partial K_o^2}{\partial c} \left[\int_{-1}^{1} \frac{(\bar{f}(\mu) - \bar{f}(-\mu))}{2} \frac{d\mu}{1 - K_o \mu} \right] \cosh K_o z$$

$$+ \frac{c}{K_o} \frac{\partial K_o^2}{\partial c} \left[\int_{-1}^{1} \frac{(\bar{f}(\mu) + \bar{f}(-\mu))}{2} \frac{d\mu}{1 - K_o \mu} \right] \sinh K_o z \qquad (21)$$

The even part of the source thus causes a discontinuity in the coefficient of the sinh $K_o z$ solution which holds far from the source while the odd part causes a discontinuity in the coefficient of cosh $K_o z$.

It is often useful to work with a solution which vanishes for $z < 0$. This solution whose asymptotic form is obtained by subtracting (20) from (19) is just (21). This can be rewritten as:

$$\rho_{as}(z) = \begin{cases} c\,\dfrac{\partial K_0^2}{\partial c}\,(\operatorname{sgn} I_+)\sqrt{|I_+|\,|I_-|}\;\dfrac{1}{2K_0}\left[e^{K_0(z+z_0)} - (\operatorname{sgn} I_+I_-)e^{-K_0(z+z_0)}\right] & (z>0) \\ 0 & (z<0) \end{cases} \tag{22}$$

where

$$I_\pm = \int_{-1}^{1} \frac{\bar{f}(\mu)\,d\mu}{1 \pm K_0\mu} \tag{23}$$

Here $\operatorname{sgn} I$ stands for the sign of I and

$$z_0 = \frac{1}{2K_0}\log\frac{|I_+|}{|I_-|} \tag{24}$$

(22) contains either $\sinh K_0(z+z_0)$ or $\cosh K_0(z+z_0)$ depending on whether $\operatorname{sgn} I_+ I_- \gtrless 0$. For simplicity we will carry through the analysis on the assumption that the sign is positive. Then

$$\rho_{as}(z) = \begin{cases} c\,\dfrac{\partial K_0^2}{\partial c}(\operatorname{sgn} I_+)\sqrt{|I_+|\,|I_-|}\;\dfrac{\sinh K_0(z+z_0)}{K_0} & (z>0) \\ 0 & (z<0) \end{cases} \tag{25}$$

Since

$$\frac{1+K_0}{1-K_0} \le \frac{I_+}{I_-} \le \frac{1-K_0}{1+K_0} \tag{26}$$

we obtain from (24) using the definition of K_0

$$-1 < c\,z_0 < 1 \tag{27}$$

The $\begin{Bmatrix}\text{upper}\\ \text{lower}\end{Bmatrix}$ bound is reached as $\bar{f}(\mu)$ approaches a δ function for $\mu = \begin{Bmatrix}-1\\ +1\end{Bmatrix}$.

(It has been mentioned that the asymptotic density is not (19) if $\bar{f}(\mu)$ is <u>actually</u> a δ function, but this is irrelevant here.) Furthermore, it is seen from (24) that if $\bar{f}(\mu)$ is always of one sign and non-vanishing only for $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right\}\mu$, z_o will be $\left\{\begin{matrix}\text{negative}\\ \text{positive}\end{matrix}\right\}$. If $\bar{f}(\mu)$ is a symmetrical function of μ, z_o will be zero.

For $c = 1$ we obtain by expansion of (23) and (24)

$$z_o = \frac{-\int_{-1}^{1} \mu\, \bar{f}(\mu)\, d\mu}{\int_{-1}^{1} \bar{f}(\mu)\, d\mu}, \quad c=1 \tag{28}$$

and (25) becomes

$$\rho_{as}(z) = \left\{\begin{matrix} -3 \int_{-1}^{1} \bar{f}(\mu)\, d\mu\, (z + z_o) \\ \\ 0 \end{matrix}\right\} \tag{29}$$

(25) also remains valid for $c > 1$. It should be noted that, in contrast to (19) and (20), (25) remains real for $c > 1$. This may be seen as follows. Putting $k_o = i\kappa_o$ we have

$$(\text{sgn } I_+)\sqrt{|I_+|\ |I_-|} = \left(\int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 + k_o^2 \mu^2}\right)\sqrt{1 + \beta^2} \tag{30}$$

where

$$\beta = k_o \frac{\int_{-1}^{1} \frac{\mu\, \bar{f}(\mu)\, d\mu}{1 + k_o^2 \mu^2}}{\int_{-1}^{1} \frac{\bar{f}(\mu)\, d\mu}{1 + k_o^2 \mu^2}} \tag{31}$$

Instead of using (24) we may determine z_o from the equations:

$$\sin 2k_o z_o = -\frac{2\beta}{1+\beta^2}$$

$$\cos 2k_o z_o = \frac{1-\beta^2}{1+\beta^2} \qquad (32)$$

and write (25) as

$$\rho_{as}(z) = \begin{cases} -c\,\dfrac{\partial k_o^2}{\partial c}\left(\displaystyle\int_{-1}^{1}\frac{-\bar{f}(\mu)\,d\mu}{1+k_o^2\mu^2}\right)\sqrt{1+\beta^2}\,\dfrac{\sin k_o(z+z_o)}{k_o} & (z>0) \\ 0 & (z<0) \end{cases} \qquad (33)$$

In simple cases it is convenient to expand $\bar{f}(\mu)$ in terms of Legendre polynomials

$$\bar{f}(\mu) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\mu)$$

$$\bar{f}(-\mu) = \sum_{\ell=0}^{\infty} (-1)^\ell a_\ell P_\ell(\mu) \qquad (34)$$

Inserting (34) into (16) and (17) and using the relation*

$$\frac{Q_\ell\left(\frac{1}{K_o}\right)}{K_o} = \frac{1}{2}\int_{-1}^{1}\frac{P_\ell(\mu)\,d\mu}{1-K_o\mu} \qquad (35)$$

(where Q_ℓ is the Legendre function of the second kind) yields

$$J = \frac{2c}{K_o}\sum_{\ell=0}^{\infty}\begin{pmatrix}1\\(-1)^\ell\end{pmatrix} a_\ell\, Q_\ell\left(\frac{1}{K_o}\right) \qquad \begin{pmatrix}(z>0)\\(z<0)\end{pmatrix} \qquad (36)$$

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* W. Magnus and F. Oberhettinger, "Special Functions of Mathematical Physics." p. 58.

Hence on remembering that

$$\frac{c}{K_o} = \frac{1}{\tanh^{-1} K_o} \tag{37}$$

(36) becomes:

$$J = 2\sum_{\ell=0}^{\infty} \begin{pmatrix} 1 \\ (-1)^{\ell} \end{pmatrix} \frac{a_{\ell}\, Q_{\ell}\left(\frac{1}{K_o}\right)}{\tanh^{-1} K_o} \qquad \begin{pmatrix} (z>0) \\ (z<0) \end{pmatrix} \tag{38}$$

Inserting the explicit forms for the Q_{ℓ} gives (with (37))

$$J = 2\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_o + \begin{pmatrix} 1 \\ -1 \end{pmatrix} a_1 \frac{(1-c)}{K_o} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_2 \left[\frac{3(1-c)}{2\,K_o^{\,2}} - \frac{1}{2}\right] \right.$$

$$\left. + \begin{pmatrix} 1 \\ -1 \end{pmatrix} a_3 \left[\frac{5(1-c)}{2\,K_o^{\,3}} + \frac{2c}{3K_o} - \frac{3}{2K_o}\right] + \cdots \right\} \qquad \begin{pmatrix} (z>0) \\ (z<0) \end{pmatrix} \tag{39}$$

In particular for an isotropic source

$$a_{\ell} = \frac{1}{2}\,\delta_{\ell o} \tag{40}$$

and

$$J = 1 \tag{41}$$

Explicit formulas for the non-asymptotic density may be obtained by inserting (5) and p(z) from (15-2) into (9). This yields

$$\begin{pmatrix} \text{aniso.} \\ \rho_{p\ell} \end{pmatrix}_{n-as} = \int_0^1 \bar{f}(\mu)e^{-z/\mu}\frac{d\mu}{\mu} + \frac{c}{2}\int_0^1 \frac{d\mu'}{\mu'} g(c,\mu') \int_{-1}^{0} \bar{f}(\mu)\frac{d\mu}{\mu}\left\{\frac{\mu\mu'}{\mu'-\mu}e^{-z/\mu'}\right\} \tag{42}$$

$$+\frac{c}{2}\int_0^1 \frac{d\mu'}{\mu'} g(c,\mu') \int_0^1 \bar{f}(\mu)\frac{d\mu}{\mu}\left\{\frac{\mu\mu'}{\mu'-\mu}\left[e^{-z/\mu'} - e^{-z/\mu}\right] - \frac{\mu\mu'}{\mu'+\mu}e^{-z/\mu}\right\} \quad (z>0)$$

$$= -\int_0^1 \bar{f}(\mu)e^{-z/\mu}\frac{d\mu}{\mu} + \frac{c}{2}\int_0^1 \frac{d\mu'}{\mu'} g(c,\mu') \int_0^1 \bar{f}(\mu)\frac{d\mu}{\mu}\left\{\frac{\mu\mu'}{\mu'+\mu}e^{+z/\mu'}\right\} \tag{43}$$

$$+\frac{c}{2}\int_0^1 \frac{d\mu'}{\mu'} g(c,\mu') \int_{-1}^{0} \bar{f}(\mu)\frac{d\mu}{\mu}\left\{\frac{\mu\mu'}{\mu'+\mu}\left[e^{-z/\mu'} - e^{-z/\mu}\right] + \frac{\mu\mu'}{\mu'-\mu}e^{-z/\mu}\right\} \quad (z<0)$$

16.2 Anisotropic Point Source

The treatment of this problem is quite similar to that of the plane source.

Let $q(\vec{r},\vec{\Omega}) = f(\vec{\Omega})\ \delta(\vec{r})$ (1)

where $\int f(\vec{\Omega})\, d\Omega = 1$ (2)

Dividing the neutron density into "direct" and indirect contributions

$$\rho^{aniso}(\vec{r}) = \rho_d(\vec{r}) + \rho_{id}(\vec{r}) \tag{3}$$

we have from (6-2)

$$\rho_d(\vec{r}) = \frac{f(\frac{\vec{r}}{r})\ e^{-r}}{r^2} \tag{4}$$

ρ_{id} is the density due to those neutrons which have suffered at least one collision, i.e., $\rho_{id}(\vec{r})$ is the density due to an effective isotropic source

$$q_{eff}(\vec{r}) = c\ \rho_d(\vec{r}) \tag{5}$$

Then if $\rho_p(|\vec{r} - \vec{r}\,'|)$ is the density at $\vec{r}$ due to a unit isotropic source at $\vec{r}\,'$ (thus ρ_p is just our point source solution) we have

$$\begin{aligned} \rho_{id}(\vec{r}) &= c \int \rho_p(|\vec{r} - \vec{r}\,'|)\, \rho_d(\vec{r}\,')\ (d\vec{r}\,') \\ &= c \int \rho_p(|\vec{r} - \vec{r}\,'|)\ \frac{f(\frac{\vec{r}\,'}{r'})\ e^{-r'}}{r'^2}\ (d\vec{r}\,') \end{aligned} \tag{6}$$

Provided $f(\vec{\Omega})$ is non-singular (i.e., does not involve delta functions) the asymptotic and non-asymptotic portions of $\rho(r)$ may be found by inserting the corresponding solutions from (14-36).

Thus:

$$\rho^{aniso}_{n-as} = f(\tfrac{\vec{r}}{r})\,\frac{e^{-r}}{r^2} + c \int p(|\vec{r} - \vec{r}\,'|)\ f(\tfrac{\vec{r}\,'}{r'})\,\frac{e^{-r'}}{r'^2}\, d\vec{r}\,' \tag{7}$$

and

$$\rho_{as}^{aniso} = c \int \rho_{as}(|\vec{r} - \vec{r}'|)\, f(\frac{\vec{r}'}{r'})\, \frac{e^{-r'}}{r'^2}\, d\vec{r}' \tag{8}$$

Inserting (14-37) and expanding the difference $|\vec{r} - \vec{r}'|$ in the exponential yields

$$\rho_{as}^{aniso} \approx - \frac{\partial \kappa_o^2}{\partial c} \frac{e^{-\kappa_o r}}{4\pi r}\, c \int e^{\kappa_o \frac{\vec{r}}{r}\cdot\vec{r}'}\, f(\frac{\vec{r}'}{r'})\, \frac{e^{-r'}}{r'^2}\, (d\vec{r}') \qquad (r \to \infty) \tag{9}$$

In particular, for the isotropic case where $f(\frac{\vec{r}}{r}) = \frac{1}{4\pi}$ we obtain

$$\rho_{as}^{iso} = - \frac{\partial \kappa_o^2}{\partial c} \frac{e^{-\kappa_o r}}{4\pi r}\, c\, \frac{\tanh^{-1}\kappa_o}{\kappa_o} = - \frac{\partial \kappa_o^2}{\partial c} \frac{e^{-\kappa_o r}}{4\pi r} \tag{10}$$

which, of course, agrees with (14-37). From (9) and (10) we see that the asymptotic solutions for anisotropic and isotropic sources merely differ by a constant factor, i.e.,

$$\frac{\rho_{as}^{aniso}}{\rho_{as}^{iso}} = \frac{\kappa_o}{\tanh^{-1}\kappa_o} \int e^{\kappa_o \frac{\vec{r}}{r}\cdot\vec{r}'}\, f(\frac{\vec{r}'}{r})\, \frac{e^{-r'}}{r'^2}\, (d\vec{r}') \tag{11}$$

Equation (11) may be put in a more useful form by expanding $f(\vec{\Omega})$ in terms of spherical harmonics. Thus if $Y_\ell^m(\vec{\Omega})$ are the orthonormal spherical harmonics

$$f(\vec{\Omega}) = \sum_{\ell m} a_{\ell m}\, Y_\ell^m(\vec{\Omega}) \tag{12}$$

and

$$a_{\ell m} = \int Y_\ell^{m*}(\vec{\Omega})\, f(\vec{\Omega})\, d\Omega \tag{13}$$

Inserting the expansion

$$e^{K_o \frac{\vec{r}}{r} \cdot \vec{r}'} = \left(\frac{\pi}{2 K_o r'}\right)^{\frac{1}{2}} \sum_{\ell=0}^{\infty} (2\ell+1)\, I_{\ell+\frac{1}{2}}(K_o r')\; P_\ell(\vec{\Omega}\cdot\vec{\Omega}') \tag{14}$$

where

$$\vec{\Omega} = \frac{\vec{r}}{r}\,; \quad \vec{\Omega}' = \frac{\vec{r}'}{r'} \tag{15}$$

gives:

$$I = \int e^{K_o \frac{\vec{r}}{r} \cdot \vec{r}'} f\left(\frac{\vec{r}'}{r'}\right) \frac{e^{-r'}}{r'^2}\, (d\vec{r}')$$

$$= \sum_{\ell=0}^{\infty} (2\ell+1) \int I_{\ell+\frac{1}{2}}(K_o r') \left(\frac{\pi}{2 K_o r'}\right)^{\frac{1}{2}} \frac{e^{-r'}}{r'^2} f(\vec{\Omega})\, P_\ell(\vec{\Omega}\cdot\vec{\Omega}')\, d\Omega'\; r'^2\, dr' \tag{16}$$

Using the addition theorem for spherical harmonics:

$$P_\ell(\vec{\Omega}\cdot\vec{\Omega}') = \frac{4\pi}{2\ell+1} \sum_{\ell m=-\ell}^{\ell} Y_\ell^{m}(\vec{\Omega})\, Y_\ell^{m*}(\vec{\Omega}') \tag{17}$$

and equation (13) gives

$$I = 4\pi \sum_{\ell m} a_{\ell m} Y_\ell^{m}(\vec{\Omega}) \int_0^{\infty} e^{-r'} I_{\ell+\frac{1}{2}}(K_o r') \left(\frac{\pi}{2 K_o r'}\right)^{\frac{1}{2}} dr' \tag{18}$$

Now it is known* that

$$\int_0^{\infty} e^{-r'} I_{\ell+\frac{1}{2}}(K_o r') \left(\frac{\pi}{2 K_o r'}\right)^{\frac{1}{2}} dr' = \frac{1}{K_o} Q_\ell\left(\frac{1}{K_o}\right) \tag{19}$$

Hence

$$I = \frac{4\pi}{K_o} \sum_{\ell m} Q_\ell\left(\frac{1}{K_o}\right) a_{\ell m} Y_\ell^{m}(\vec{\Omega}) \tag{20}$$

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* G. N. Watson, "Theory of Bessel Functions," Cambridge, 1945. p. 387.

and therefore (11) becomes

$$\frac{\rho_{as}^{aniso}}{\rho_{as}^{iso}} = \sum_{\ell m} 4\pi \, a_{\ell m} \frac{Q_\ell\left(\frac{1}{K_o}\right)}{\tanh^{-1} K_o} \, Y_\ell^m(\vec{\Omega}) \tag{21}$$

In particular for an isotropic source the only non-vanishing terms are $\ell = 0$, $m = 0$. Then the right hand side of (21) becomes (as expected):

$$\frac{Q_o\left(\frac{1}{K_o}\right)}{\tanh^{-1} K_o} = 1 \tag{22}$$

Appendix A

The Functions $E_n(x) = \int_1^\infty e^{-xu} u^{-n} du$

The functions $E_n(x) = \int_1^\infty e^{-xu} u^{-n} du$ have been defined by Schloemilch,[1] and have been extensively used by Schwarzschild,[2] Eddington,[3] Hopf[4] and others. Their Fourier and Laplace transforms have been discussed by Placzek.[5] An extensive table of integrals containing these functions has been given by Le Caine.[6]

In the following we give a few useful formulae for the functions $E_n^*(x)$ as well as a table of the first four of the functions taken from the more complete tables, extending up to $n = 20$, prepared by the Mathematical Tables Project.[5]

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(1) Zeitschrift für Math. und Phys. 4, 390, 1859.

(2) Gött. Nachr. Math. Phys. Klasse, 1906, p. 41.
Berliner Berichte Math. Phys. Klasse, 1914, p. 1183.

(3) "The Internal Constitution of the Stars," Cambridge, 1926.

(4) "Mathematical Problems of Radiative Equilibrium," Cambridge Tracts No. 31, 1934.

(5) "The Functions $E_n(x)$," National Research Council of Canada, Atomic Energy Project Report MT-1.

(6) "A Table of Integrals Involving the Functions $E_n(x)$," National Research Council of Canada, Atomic Energy Project Report MT-131.

* Since most applications deal with positive integral values of n and real x, we restrict ourselves to this case, although most of the relations given have a wider range of validity.

The Functions $E_n(x)$ for Real Positive Argument and Positive Integral n

1) Definition:

$$E_n(x) = \int_1^\infty e^{-xu}\, u^{-n}\, du = \int_o^1 \mu^{n-2}\, e^{-x/\mu}\, d\mu \tag{1}$$

in particular:

$$E_o(x) = \frac{e^{-x}}{x} \tag{1a}$$

$$E_1(x) = -\,Ei(-x) = \int_x^\infty e^{-u}\,\frac{du}{u} \tag{1b}$$

$$E_n(x) = Q_n(x)\, e^{-x} + \frac{(-x)^{n-1}}{(n-1)!}\, E_1(x), \qquad n > 1 \tag{1c}$$

with

$$Q_n(x) = \frac{1}{(n-1)!} \sum_{m=o}^{n-2} (n-m-2)!\,(-x)^m \tag{1d}$$

2) Recurrence Relations:

$$E_n(x) = \int_x^\infty E_{n-1}(x')\,dx' \tag{2}$$

$$E_n'(x) = -\,E_{n-1}(x) \tag{3}$$

$$E_n(x) = \frac{1}{n-1}\left\{ e^{-x} - x\, E_{n-1}(x) \right\}, \qquad n > 1 \tag{4}$$

3) Expansions:

(a) The following expansion is equivalent to the definition (1):

$$E_n(x) = \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty}{}' \frac{(-x)^m}{m!(n-1-m)} + (-1)^n \frac{x^{n-1}}{(n-1)!} \left\{ \log \gamma x - A_n \right\}, \quad n > 0$$

$$\log \gamma = .577216 \qquad A_1 = 0, \quad A_n = \sum_{m=1}^{n-1} \frac{1}{m}, \quad n > 1 \tag{5}$$

(b) Asymptotic Expansions $(x \gg 1)$

$$E_n(x) = \frac{e^{-x}}{x} \left\{ 1 - \frac{n}{x} + \frac{n(n+1)}{x^2} - \frac{n(n+1)(n+2)}{x^3} + \dots \right\} \qquad n > 0 \tag{6}$$

Considerably more useful is the following expansion, due to G. Blanch[5]:

$$E_n(x) = \frac{e^{-x}}{x+m} \left[1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} \right] + R(n,x) \tag{7}$$

where

$$R(x,n) = n \int_1^{\infty} \frac{e^{-xu}(-24x^3u^3 + 58nx^2u^2 - 22n^2xu + n^3)}{u^n(xu+n)^8}\, du \tag{8}$$

4) Inequalities:

$$E_n(x) \leq \frac{n}{n-1} E_{n+1}(x), \qquad n > 1 \tag{9}$$

$$\frac{e^{-x}}{x+n} < E_n(x) \leq \frac{e^{-x}}{x+n-1} \qquad n \geq 1 \tag{10}$$

$$E_n^2(x) < E_{n-1}(x)\, E_{n+1}(x) \tag{11}$$

$$\frac{d}{dx} \left(\frac{E_{n+1}(x)}{E_n(x)} \right) > 0 \tag{12}$$

$$\frac{d}{dx} \frac{E_n(x-a)}{E_n(x)} < 0, \qquad 0 < a < x \tag{13}$$

TABLE 22

TABLE OF THE FUNCTIONS $E_n^*(x)$

x	$E_0(x)$	$E_1(x)$	$E_2(x)$	$E_3(x)$	$E_4(x)$
.00	∞	∞	1.0000000	.5000000	.3333333
.01	99.0049834	4.0379296	0.9496705	.4902766	.3283824
.02	49.0099337	3.3547078	.9131045	.4809683	.3235264
.03	32.3481844	2.9591187	.8816720	.4719977	.3187619
.04	24.0197360	2.6812637	.8535389	.4633239	.3140855
.05	19.0245885	2.4678985	.8278345	.4549188	.3094945
.06	15.6960756	2.2953069	.8040461	.4467609	.3049863
.07	13.3199117	2.1508382	.7818352	.4388327	.3005585
.08	11.5389543	2.0269410	.7609611	.4311197	.2962089
.09	10.1547909	1.9187448	.7412442	.4236096	.2919354
.10	9.0483742	1.8229240	.7225450	.4162915	.2877361
.11	8.1439467	1.7371067	.7047524	.4091557	.2836090
.12	7.3910036	1.6595418	.6877754	.4021937	.2795524
.13	6.7545802	1.5888993	.6715385	.3953977	.2755646
.14	6.2097017	1.5241457	.6559778	.3887607	.2716439
.15	5.7380532	1.4644617	.6410387	.3822761	.2677889
.16	5.3258987	1.4091867	.6266739	.3759380	.2639979
.17	4.9627342	1.3577806	.6128421	.3697408	.2602696
.18	4.6403901	1.3097961	.5995069	.3636795	.2566026
.19	4.3524165	1.2648584	.5866360	.3577491	.2529956
.20	4.0936538	1.2226505	.5742006	.3519453	.2494472
.21	3.8599250	1.1829020	.5621748	.3462638	.2459563
.22	3.6478127	1.1453801	.5505352	.3407005	.2425216
.23	3.4544939	1.1098831	.5392605	.3352518	.2391419
.24	3.2776161	1.0762354	.5283314	.3299142	.2358162
.25	3.1152031	1.0442826	.5177301	.3246841	.2325432
.26	2.9655830	1.0138887	.5074405	.3195585	.2293221
.27	2.8273315	0.9849331	.4974476	.3145343	.2261517
.28	2.6992276	.9573083	.4877374	.3096086	.2230311
.29	2.5802192	.9309182	.4782973	.3047787	.2199593
.30	2.4693941	.9056767	.4691152	.3000418	.2169352
.31	2.3659579	.8815057	.4601802	.2953956	.2139581
.32	2.2692157	.8583352	.4514818	.2908374	.2110270
.33	2.1785568	.8361012	.4430104	.2863652	.2081411
.34	2.0934421	.8147456	.4347568	.2819765	.2052994
.35	2.0133945	.7942154	.4267127	.2776693	.2025013
.36	1.9379898	.7744622	.4188699	.2734416	.1997458
.37	1.8668495	.7554414	.4112210	.2692913	.1970322
.38	1.7996353	.7371121	.4037588	.2652165	.1943597
.39	1.7360433	.7194367	.3964766	.2612155	.1917276

* See also page A 11.

x	E_0	E_1	E_2	E_3	E_4
.40	1.6758001	.7023801	.3893680	.2572864	.1891352
.41	1.6186591	.6859103	.3824270	.2534276	.1865816
.42	1.5643972	.6699973	.3756479	.2496373	.1840664
.43	1.5128118	.6546134	.3690253	.2459141	.1815887
.44	1.4637191	.6397328	.3625540	.2422563	.1791479
.45	1.4169514	.6253313	.3562291	.2386625	.1767433
.46	1.3723558	.6113865	.3500458	.2351313	.1743744
.47	1.3297921	.5978774	.3439999	.2316612	.1720405
.48	1.2891321	.5847843	.3380869	.2282508	.1697410
.49	1.2502579	.5720888	.3323029	.2248990	.1674753
.50	1.2130613	.5597736	.3266439	.2216044	.1652428
.51	1.1774423	.5478224	.3211062	.2183657	.1630430
.52	1.1433087	.5362198	.3156863	.2151818	.1608753
.53	1.1105754	.5249515	.3103807	.2120516	.1587392
.54	1.0791634	.5140039	.3051862	.2089739	.1566341
.55	1.0489997	.5033641	.3000996	.2059475	.1545596
.56	1.0200162	.4930200	.2951179	.2029715	.1525150
.57	0.9921499	.4829600	.2902382	.2000448	.1505000
.58	0.9653420	.4731734	.2854578	.1971664	.1485139
.59	0.9395378	.4636498	.2807739	.1943353	.1465565
.60	0.9146861	.4543795	.2761839	.1915506	.1446271
.61	0.8907391	.4453531	.2716855	.1888114	.1427253
.62	0.8676523	.4365619	.2672761	.1861166	.1408507
.63	0.8453838	.4279973	.2629535	.1834656	.1390028
.64	0.8238944	.4196516	.2587154	.1808573	.1371813
.65	0.8031473	.4115170	.2545597	.1782910	.1353855
.66	0.7831081	.4035863	.2504844	.1757658	.1336153
.67	0.7637441	.3958526	.2464874	.1732810	.1318701
.68	0.7450250	.3883092	.2425667	.1708358	.1301495
.69	0.7269218	.3809500	.2387206	.1684294	.1284533
.70	0.7094076	.3737688	.2349471	.1660612	.1267808
.71	0.6924566	.3667600	.2312446	.1637303	.1251319
.72	0.6760448	.3599179	.2276114	.1614360	.1235061
.73	0.6601493	.3532374	.2240457	.1591778	.1219031
.74	0.6447485	.3467133	.2205461	.1569549	.1203224
.75	0.6298221	.3403408	.2171109	.1547667	.1187638
.76	0.6153506	.3341153	.2137388	.1526125	.1172270
.77	0.6013157	.3280323	.2104282	.1504917	.1157115
.78	0.5877000	.3220876	.2071777	.1484037	.1142170
.79	0.5744871	.3162770	.2039860	.1463479	.1127433

x	E_0	E_1	E_2	E_3	E_4
.80	0.5616612	.3105966	.2008517	.1443238	.1112900
.81	0.5492075	.3050425	.1977736	.1423307	.1098567
.82	0.5371118	.2996112	.1947504	.1403681	.1084433
.83	0.5253606	.2942992	.1917810	.1384355	.1070493
.84	0.5139411	.2891029	.1888641	.1365324	.1056744
.85	0.5028411	.2840193	.1859986	.1346581	.1043185
.86	0.4920489	.2790451	.1831833	.1328122	.1029812
.87	0.4815535	.2741773	.1804173	.1309943	.1016622
.88	0.4713442	.2694130	.1776994	.1292037	.1003612
.89	0.4614110	.2647495	.1750287	.1274401	.0990780
.90	0.4517441	.2601839	.1724041	.1257030	.0978123
.91	0.4423343	.2557138	.1698247	.1239919	.0965639
.92	0.4331729	.2513364	.1672895	.1223063	.0953324
.93	0.4242513	.2470495	.1647977	.1206459	.0941177
.94	0.4155615	.2428506	.1623482	.1190102	.0929194
.95	0.4070958	.2387375	.1599404	.1173988	.0917374
.96	0.3988468	.2347080	.1575732	.1158113	.0905713
.97	0.3908073	.2307599	.1552459	.1142472	.0894211
.98	0.3829705	.2268912	.1529578	.1127063	.0882863
.99	0.3753300	.2230998	.1507079	.1111880	.0871669
1.00	0.3678794	.2193839	.1484955	.1096920	.0860625
1.01	.3606129	.2157416	.1463199	.1082179	.0849730
1.02	.3535245	.2121711	.1441804	.1067654	.0838981
1.03	.3466087	.2086706	.1420763	.1053342	.0828376
1.04	.3398603	.2052384	.1400068	.1039238	.0817913
1.05	.3332740	.2018728	.1379713	.1025339	.0807590
1.06	.3268451	.1985723	.1359691	.1011643	.0797406
1.07	.3205687	.1953354	.1339996	.0998145	.0787357
1.08	.3144403	.1921605	.1320622	.0984842	.0777442
1.09	.3084555	.1890461	.1301562	.0971731	.0767659
1.10	.3026101	.1859909	.1282811	.0958809	.0758007
1.11	.2969000	.1829935	.1264362	.0946074	.0748483
1.12	.2913212	.1800525	.1246210	.0933521	.0739085
1.13	.2858701	.1771666	.1228350	.0921149	.0729812
1.14	.2805430	.1743347	.1210775	.0908953	.0720661
1.15	.2753363	.1715554	.1193481	.0896932	.0711632
1.16	.2702467	.1688275	.1176462	.0885083	.0702722
1.17	.2652709	.1661500	.1159714	.0873402	.0693930
1.18	.2604057	.1635217	.1143231	.0861888	.0685253
1.19	.2556481	.1609416	.1127008	.0850537	.0676691

x	E_0	E_1	E_2	E_3	E_4
1.20	.2509952	.1584084	.1111041	.0839347	.0668242
1.21	.2464440	.1559213	.1095325	.0828315	.0659904
1.22	.2419919	.1534792	.1079855	.0817439	.0651675
1.23	.2376362	.1510812	.1064627	.0806717	.0643555
1.24	.2333744	.1487262	.1049637	.0796146	.0635540
1.25	.2292038	.1464134	.1034881	.0785723	.0627631
1.26	.2251222	.1441418	.1020353	.0775447	.0619825
1.27	.2211273	.1419106	.1006051	.0765316	.0612122
1.28	.2172166	.1397190	.0991970	.0755326	.0604519
1.29	.2133882	.1375660	.0978106	.0745476	.0597015
1.30	.2096398	.1354510	.0964455	.0735763	.0589609
1.31	.2059695	.1333730	.0951015	.0726186	.0582299
1.32	.2023752	.1313313	.0937780	.0716742	.0575085
1.33	.1988551	.1293252	.0924747	.0707429	.0567964
1.34	.1954072	.1273540	.0911913	.0698246	.0560936
1.35	.1920298	.1254168	.0899275	.0689191	.0553998
1.36	.1887212	.1235131	.0886829	.0680260	.0547151
1.37	.1854795	.1216422	.0874571	.0671453	.0540393
1.38	.1823033	.1198033	.0862499	.0662768	.0533722
1.39	.1791909	.1179959	.0850610	.0654203	.0527137
1.40	.1761407	.1162193	.0838899	.0645755	.0520637
1.41	.1731513	.1144729	.0827365	.0637424	.0514222
1.42	.1702211	.1127561	.0816004	.0629207	.0507889
1.43	.1673489	.1110683	.0804813	.0621104	.0501637
1.44	.1645332	.1094089	.0793789	.0613111	.0495466
1.45	.1617726	.1077774	.0782930	.0605227	.0489374
1.46	.1590659	.1061733	.0772233	.0597452	.0483361
1.47	.1564119	.1045959	.0761694	.0589782	.0477425
1.48	.1538092	.1030449	.0751313	.0582217	.0471565
1.49	.1512568	.1015196	.0741085	.0574755	.0465780
1.50	.1487534	.1000196	.0731008	.0567395	.0460070
1.51	.1462980	.0985444	.0721080	.0560135	.0454432
1.52	.1438894	.0970935	.0711298	.0552973	.0448867
1.53	.1415266	.0956664	.0701660	.0545908	.0443372
1.54	.1392085	.0942628	.0692164	.0538939	.0437948
1.55	.1369342	.0928821	.0682807	.0532064	.0432593
1.56	.1347026	.0915240	.0673587	.0525283	.0427307
1.57	.1325129	.0901879	.0664502	.0518592	.0422087
1.58	.1303640	.0888736	.0655549	.0511992	.0416935
1.59	.1282551	.0875805	.0646726	.0505481	.0411847

x	E_0	E_1	E_2	E_3	E_4
1.60	.1261853	.0863083	.0638032	.0499057	.0406825
1.61	.1241538	.0850567	.0629464	.0492720	.0401866
1.62	.1221597	.0838251	.0621020	.0486467	.0396970
1.63	.1202022	.0826134	.0612698	.0480299	.0392136
1.64	.1182805	.0814210	.0604497	.0474213	.0387364
1.65	.1163939	.0802476	.0596413	.0468209	.0382652
1.66	.1145416	.0790930	.0588446	.0462284	.0377999
1.67	.1127228	.0779567	.0580594	.0456439	.0373406
1.68	.1109369	.0768384	.0572854	.0450672	.0368870
1.69	.1091832	.0757378	.0565226	.0444982	.0364392
1.70	.1074609	.0746546	.0557706	.0439367	.0359970
1.71	.1057695	.0735885	.0550294	.0433827	.0355604
1.72	.1041082	.0725392	.0542988	.0428361	.0351293
1.73	.1024765	.0715063	.0535786	.0422967	.0347037
1.74	.1008738	.0704895	.0528686	.0417645	.0342834
1.75	.0992994	.0694887	.0521687	.0412393	.0338684
1.76	.0977528	.0685034	.0514788	.0407211	.0334586
1.77	.0962333	.0675335	.0507986	.0402097	.0330539
1.78	.0947405	.0665787	.0501281	.0397051	.0326544
1.79	.0932738	.0656386	.0494670	.0392071	.0322598
1.80	.0918327	.0647131	.0488153	.0387157	.0318702
1.81	.0904167	.0638019	.0481727	.0382308	.0314855
1.82	.0890251	.0629047	.0475392	.0377522	.0311056
1.83	.0876577	.0620213	.0469146	.0372800	.0307304
1.84	.0863138	.0611515	.0462987	.0368139	.0303599
1.85	.0849931	.0602950	.0456915	.0363540	.0299941
1.86	.0836950	.0594515	.0450928	.0359001	.0296328
1.87	.0824191	.0586210	.0445024	.0354521	.0292761
1.88	.0811649	.0578031	.0439203	.0350100	.0289238
1.89	.0799322	.0569976	.0433463	.0345737	.0285759
1.90	.0787203	.0562044	.0427803	.0341430	.0282323
1.91	.0775290	.0554231	.0422222	.0337180	.0278930
1.92	.0763578	.0546537	.0416718	.0332986	.0275579
1.93	.0752063	.0538959	.0411291	.0328846	.0272270
1.94	.0740742	.0531495	.0405938	.0324759	.0269002
1.95	.0729611	.0524144	.0400660	.0320727	.0265775
1.96	.0718665	.0516903	.0395455	.0316746	.0262587
1.97	.0707903	.0509770	.0390322	.0312817	.0259440
1.98	.0697319	.0502744	.0385259	.0308939	.0256331
1.99	.0686912	.0495823	.0380267	.0305112	.0253261

x	E_0	E_1	E_2	E_3	E_4
2.0	6.76676(-2)	4.89005(-2)	3.75343(-2)	3.01334(-2)	2.50228(-2)
2.1	5.83126	4.26143	3.29663	2.66136	2.21893
2.2	5.03651	3.71911	2.89827	2.35207	1.96859
2.3	4.35908	3.25023	2.55036	2.08002	1.74728
2.4	3.77991	2.84403	2.24613	1.84054	1.55150
2.5	3.28340	2.49149	1.97977	1.62954	1.37822
2.6	2.85668	2.18502	1.74630	1.44349	1.22476
2.7	2.48909	1.91819	1.54145	1.27932	1.08879
2.8	2.17179	1.68553	1.36152	1.13437	0.96826
2.9	1.89735	1.48240	1.20336	1.00629	0.86136
3.0	1.65957	1.30484	1.06419	0.89306	0.76650
3.1	1.45320	1.14944	0.94165	0.79290	0.68231
3.2	1.27382	1.01330	0.83366	0.70425	0.60754
3.3	11.17672(-3)	8.93904(-3)	7.38433(-3)	6.25744(-3)	5.41120(-3)
3.4	9.81567	7.89097	6.54396	5.56190	4.82093
3.5	8.62782	6.97014	5.80189	4.94538	4.29619
3.6	7.58992	6.16041	5.14623	4.39865	3.82953
3.7	6.68203	5.44782	4.56658	3.91360	3.41440
3.8	5.88705	4.82025	4.05383	3.48310	3.04500
3.9	5.19023	4.26715	3.60004	3.10087	2.71618
4.0	4.57891	3.77935	3.19823	2.76136	2.42340
4.1	4.04212	3.34888	2.84226	2.45969	2.16264
4.2	3.57038	2.96876	2.52678	2.19156	1.93034
4.3	3.15548	2.63291	2.24704	1.95315	1.72334
4.4	2.79030	2.33601	1.99890	1.74110	1.53883
4.5	2.46867	2.07340	1.77869	1.55244	1.37434
4.6	2.18518	1.84101	1.58321	1.38454	1.22765
4.7	1.93517	1.63525	1.40960	1.23507	1.09682
4.8	1.71453	1.45299	1.25538	1.10197	0.98010
4.9	1.51971	1.29148	1.11831	0.98342	0.87594
5.0	1.34759	1.14830	0.99647	0.87780	0.78298
5.1	1.19544	1.02130	0.88812	0.78368	0.70000
5.2	10.6088 (-4)	9.0862 (-4)	7.9173 (-4)	6.9978 (-4)	6.2590 (-4)
5.3	9.4181	8.0861	7.0597	6.2498	5.5974
5.4	8.3640	7.1980	6.2964	5.5827	5.0064
5.5	7.4305	6.4093	5.6168	4.9877	4.4784
5.6	6.6033	5.7084	5.0116	4.4569	4.0067
5.7	5.8701	5.0855	4.4725	3.9832	3.5852
5.8	5.2199	4.5316	3.9922	3.5604	3.2084
5.9	4.6431	4.0390	3.5641	3.1830	2.8716

Note: The figures in parentheses indicate the power of ten by which the numbers alongside and below in the same column are to be multiplied.

x	E_0	E_1	E_2	E_3	E_4
6.0	4.1313(-4)	3.6008(-4)	3.1826(-4)	2.8460(-4)	2.5704(-4)
6.1	3.6768	3.2109	2.8424	2.5451	2.3012
6.2	3.2733	2.8638	2.5390	2.2763	2.0603
6.3	2.9148	2.5547	2.2683	2.0362	1.8449
6.4	2.5962	2.2795	2.0269	1.8217	1.6522
6.5	2.3130	2.0343	1.8115	1.6300	1.4798
6.6	2.0612	1.8158	1.6192	1.4586	1.3256
6.7	1.8372	1.6211	1.4475	1.3055	1.1875
6.8	1.6379	1.4476	1.2942	1.1685	1.0639
6.9	1.4606	1.2928	1.1573	1.0461	0.9533
7.0	1.3027	1.1548	1.0351	0.9366	0.8543
7.1	1.1621	1.0317	0.9259	0.8386	0.7656
7.2	10.3692(-5)	9.2188(-5)	8.2831(-5)	7.5100(-5)	6.8622(-5)
7.3	9.2540	8.2387	7.4112	6.7261	6.1511
7.4	8.2602	7.3640	6.6319	6.0247	5.5142
7.5	7.3745	6.5831	5.9353	5.3970	4.9437
7.6	6.5849	5.8859	5.3125	4.8352	4.4326
7.7	5.8809	5.2633	4.7556	4.3323	3.9747
7.8	5.2530	4.7072	4.2576	3.8821	3.5644
7.9	4.6930	4.2104	3.8122	3.4790	3.1967
8.0	4.1933	3.7666	3.4138	3.1181	2.8672
8.1	3.7474	3.3700	3.0573	2.7949	2.5719
8.2	3.3494	3.0155	2.7384	2.5054	2.3071
8.3	2.9942	2.6986	2.4530	2.2461	2.0698
8.4	2.6770	2.4154	2.1975	2.0138	1.8570
8.5	2.3937	2.1621	1.9689	1.8057	1.6662
8.6	2.1408	1.9356	1.7642	1.6192	1.4952
8.7	1.9148	1.7331	1.5810	1.4521	1.3418
8.8	1.7129	1.5519	1.4169	1.3024	1.2042
8.9	1.5325	1.3898	1.2700	1.1682	1.0808
9.0	1.3712	1.2447	1.1384	1.0479	0.9701
9.1	1.2271	1.1150	1.0205	0.9400	0.8708
9.2	10.9825(-6)	9.9881(-6)	9.1492(-6)	8.4335(-6)	7.8169(-6)
9.3	9.8306	8.9485	8.2033	7.5668	7.0177
9.4	8.8004	8.0179	7.3558	6.7896	6.3006
9.5	7.8791	7.1848	6.5965	6.0927	5.6571
9.6	7.0551	6.4388	5.9160	5.4677	5.0797
9.7	6.3179	5.7709	5.3061	4.9071	4.5614
9.8	5.6583	5.1727	4.7595	4.4044	4.0963
9.9	5.0681	4.6369	4.2695	3.9533	3.6788
10.0	4.5400	4.1570	3.8302	3.5488	3.3041

Note: The figures in parentheses indicate the power of ten by which the numbers alongside and below in the same column are to be multiplied.

x	$E_2(x) - x \log_e x$	$E_3(x) + \frac{x^2}{2} \log_e \frac{x}{2}$	x	$E_2(x) - x \log_e x$
.00	1.0000000	.5000000	.25	.8643037
.01	0.9957222	.4900463	.26	.8576797
.02	.9913450	.4801859	.27	.8509676
.03	.9868687	.4704197	.28	.8441678
.04	.9822939	.4607488	.29	.8372808
.05	.9776211	.4511742	.30	.8303071
.06	.9728508	.4416967	.31	.8232469
.07	.9679834	.4323175	.32	.8161007
.08	.9630194	.4230374	.33	.8088690
.09	.9579593	.4138574	.34	.8015521
.10	.9528035	.4047785	.35	.7941504
.11	.9475526		.36	.7866644
.12	.9422071		.37	.7790943
.13	.9367672		.38	.7714407
.14	.9312336		.39	.7637039
.15	.9256067		.40	.7558843
.16	.9198870		.41	.7479823
.17	.9140748		.42	.7399982
.18	.9081706		.43	.7319324
.19	.9021750		.44	.7237854
.20	.8960882		.45	.7155575
.21	.8899109		.46	.7072491
.22	.8836433		.47	.6988605
.23	.8772860		.48	.6903921
.24	.8708393		.49	.6818443
			.50	.6732175

Appendix B

Relations Between Point, Plane, and Shell Source Problems for the Infinite Isotropic Medium

The basis of these relations is the linearity of the equations for the neutron density. The density due to a distribution $q_s(\vec{r}\,')$ of point sources distributed over a surface S is just the sum of the densities of each.

i.e.,
$$\rho(\vec{r}) = \int_S \rho_p(|\vec{r} - \vec{r}\,'|)\, q_s(\vec{r}\,')\, dS \tag{1}$$

where $\rho_p(|\vec{r}|)$ is the density due to a unit isotropic point source at the origin.

Hence, on choosing cylindrical coordinates, the density due to a unit isotropic plane source lying on the plane $z = 0$ is;

$$\rho_{p\ell}(z) = \int_0^{\infty} r\, dr \int_0^{2\pi} d\phi\, \rho_p\left(\sqrt{z^2 + r^2}\right) \tag{2}$$

Let
$$R = \sqrt{z^2 + r^2} \tag{3}$$

Then
$$\rho_{p\ell}(z) = 2\pi \int_{|z|}^{\infty} \rho_p(R)\, R dR \tag{4}$$

On differentiating we obtain the inverse relation

$$\rho_p(r) = -\frac{1}{2\pi r} \frac{d}{dr} \rho_{p\ell}(r) \tag{5}$$

The density due to a unit shell source at $r = a$ is:

$$\rho_{sh}(\vec{r},a) = \int_{r'=a} \rho_p(|\vec{r} - \vec{r}'|) \; \frac{dS'}{4\pi a^2} \tag{6}$$

$$= \int_0^{\pi} \rho_p(\sqrt{r^2 + a^2 - 2ra\cos\theta}\;) \; \frac{d(\cos\;\theta)}{2}$$

Here θ is the angle between $\vec{r}$ and $\vec{r}'$. Introducing R (the argument of ρ_p) as variable of integration gives

$$\rho_{sh}(\vec{r},a) = \int_{|r-a|}^{|r+a|} \frac{\rho_p(R)\; RdR}{2ra} \tag{7}$$

Comparing with (4) we see

$$\rho_{sh}(\vec{r},a) = \frac{1}{4\pi ra} \left\{ \rho_{p\ell}(|r - a|) - \rho_{p\ell}(|r + a|) \right\} \tag{8}$$

Relations between the currents $\vec{j}_p$, $\vec{j}_{p\ell}$, $\vec{j}_{sh}$ due to unit isotropic point, plane, and shell sources may be obtained similarly using the linearity. Thus the current due to a surface source $q_s(\vec{r}')$ is:

$$\vec{j}(\vec{r}) = \int_s \vec{j}_p(\vec{r} - \vec{r}')\; q_s(\vec{r}')\; dS' \tag{9}$$

The current due to an isotropic point source is radially directed and a function only of the radial distance, i.e.,

$$\vec{j}_p(\vec{r}) = \frac{\vec{r}}{r}\, j_p(r) \tag{10}$$

Hence the current of a unit plane source is:

$$\vec{j}_{p\ell}(z) = \int_0^{\infty} r\; dr \int_0^{2\pi} d\phi \;\; \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|}\; j_p\;(\sqrt{z^2 + r^2}\;) \tag{11}$$

By symmetry this vector has only a component in the z direction. This component is:

$$j_{p\ell}(z) = 2\pi z \int_{|z|}^{\infty} j_p(R)\, dR \tag{12}$$

Differentiating gives the relation

$$j_p(r) = -\frac{1}{2\pi} \frac{d}{dr} \left(\frac{j_{p\ell}(r)}{r}\right) \tag{13}$$

The current due to a shell source at $r = a$ is:

$$\vec{j}_{sh}(\vec{r},a) = \int_{r'=a} \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, j_p(|\vec{r} - \vec{r}\,'|)\, \frac{dS'}{4\pi a^2} \tag{14}$$

But

$$\frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, j_p(|\vec{r} - \vec{r}\,'|) = -\nabla \int_{|r-r'|}^{\infty} j_p(R')\, dR' \tag{15}$$

Hence, on comparing with (12),

$$\begin{aligned} \vec{j}_{sh}(\vec{r},a) &= -\nabla \frac{1}{4\pi a^2} \int_{r'=a} \frac{j_{p\ell}(|\vec{r} - \vec{r}\,'|)}{2\pi\, |\vec{r} - \vec{r}\,'|}\, dS' \\ &= -\nabla \frac{1}{4\pi} \int_{-1}^{1} \frac{j_{p\ell}\left(\sqrt{r^2 + a^2 - 2ar\mu}\right)}{\sqrt{r^2 + a^2 - 2ar\mu}}\, d\mu \end{aligned} \tag{16}$$

Introducing $R = \sqrt{r^2 + a^2 - 2ar\mu}$ as variable of integration gives:

$$\vec{j}_{sh}(\vec{r},a) = -\nabla \frac{1}{4\pi ar} \int_{|r-a|}^{|r+a|} j_{p\ell}(R)\, dR \tag{17}$$

Thus $\vec{j}_{sh}(\vec{r},a)$ is radially directed. On writing

$$\vec{j}_{sh}(\vec{r},a) = \frac{\vec{r}}{r}\, j_{sh}(r,a) \tag{18}$$

we find from (17)

$$4\pi r^2 \, j_{sh}(r,a) = \frac{1}{a}\left\{ \int_{|r-a|}^{|r+a|} j_{p\ell}(z)dz - r\left[j_{p\ell}(|r+a|) - j_{p\ell}(|r-a|) \right] \right\} \qquad (19)$$

Appendix C

Transformation of the Isotropic Point Source Solution*

1. $c<1$

From (14-29) we have that the point source solution is:

$$\rho(r) = \frac{I(r)}{2\pi r} \tag{1}$$

where

$$I(r) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi(k,r)\, dk \tag{2}$$

$$\psi(k,r) = \frac{\tan^{-1}k\ e^{ikr}}{1 - c\tan^{-1}k/k}$$

$$\text{Since } \tan^{-1}k = \frac{1}{i}\tanh^{-1}ik = \frac{1}{2i}\log\frac{1+ik}{1-ik} \tag{3}$$

ψ has branch points at $k = i$ and $-i$. It is therefore single-valued in the k plane cut by branch cuts from $+i$ to $+i\infty$ and $-i$ to $-i\infty$. We choose that branch of the log function for which $\log 1 = 0$. The only other possible singularities of ψ are poles arising from zeros of

$$f(k) = 1 - c\,\frac{\tan^{-1}k}{k} \tag{4}$$

$f(k)$ being an even function of k has only an even number of zeros which occur in pairs $(k_o, -k_o)$. Hence to investigate zeros we can limit ourselves to those zeros which lie within the contour C (Figure 34).

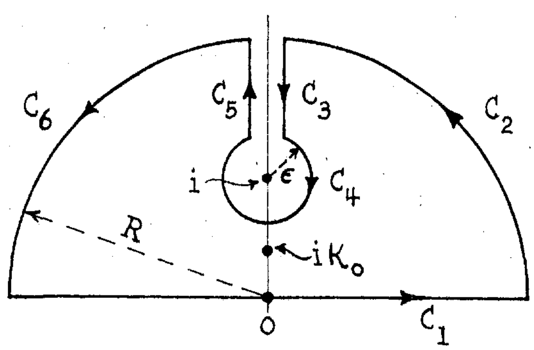

Fig. 34

* The discussion here follows closely that given by Placzek and Volkoff in Canadian Report MT-4.

We will now prove that $f(k)$ has but a single zero inside the contour C given by $k = i\kappa_o$ where κ_o is the positive root of

$$c = \frac{\kappa_o}{\tanh^{-1}\kappa_o} \tag{5}$$

To prove this we use the theorem* that, since $f(k)$ is analytic on and within C, the number of the zeros within C is given by $\frac{1}{2\pi}$ times the change in its argument on going around C in the positive direction.

The argument of $f(k)$ on C_1 (the real axis) is zero. Since $f(k) \to 1$ as $|k| \to \infty$, the argument of $f(k)$ along C_2 and $C_6 \to 0$ as $R \to \infty$. Along C_3 and C_5 we have $k = is$ (s real) and

$$\tan^{-1}k = \frac{1}{2i}\log\frac{(1 + ik)}{1 - ik} = \frac{1}{2i}\log\frac{1 - s}{1 + s} \tag{6}$$

$$\text{On } \begin{Bmatrix} C_3 \\ C_5 \end{Bmatrix} \text{ we have } s = \left\{1 - |1 - s|\, e^{\pm i\pi}\right\} \tag{7}$$

Hence

$$\begin{aligned} \tan^{-1}k &= \frac{1}{2i}\log\left\{\frac{|1 - s|\, e^{\pm i\pi}}{1 + s}\right\} = \frac{1}{2i}\log\frac{s - 1}{s + 1} \pm \frac{\pi}{2} \\ &= i\tanh^{-1}\frac{1}{s} \pm \frac{\pi}{2} \qquad \text{on } \begin{Bmatrix} C_3 \\ C_5 \end{Bmatrix} \end{aligned} \tag{8}$$

and

$$f(k) = 1 - \frac{c\tanh^{-1}\frac{1}{s}}{s} \pm \frac{i c \pi}{2s} \qquad \text{on } \begin{Bmatrix} C_3 \\ C_5 \end{Bmatrix} \tag{9}$$

On C_3 s goes from R to $1 + \epsilon$. The imaginary part of f then increases continuously from $\frac{c\pi}{2R}$ $(\to 0)$ to $\frac{c\pi}{2(1 + \epsilon)}$. The real part of f continuously decreases from $1 - \frac{c}{R}\tanh^{-1}\frac{1}{R}$ $(\to 1)$ to $1 - \frac{c\tanh^{-1}\frac{1}{1+\epsilon}}{1 + \epsilon}$ $(\to -\infty$ as $\epsilon \to 0)$.

Thus the change in the argument of $f(k)$ on $C_3 \to \pi$ as $R \to \infty$ and $\epsilon \to 0$.

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* E. C. Titchmarsh, "The Theory of Functions," Second Edition.

Now the real part of $f(k)$ on $C_4 \to -\infty$ as $\epsilon \to 0$ while the imaginary part remains finite. Therefore the change in the argument of $f(k)$ on C_4 as $\epsilon \to 0$ is zero.

The same discussion as for C_3 shows that

$$\lim_{\substack{R \to \infty \\ \epsilon \to 0}} \Delta_{C_5} \arg f(k) = \pi \tag{10}$$

Combining we see

$$\Delta_C \arg f(k) = 2\pi \tag{11}$$

Hence there is one and only one zero of $f(k)$ within C.

Therefore, since $\psi(k,r)$ is analytic within and on C except for a simple pole at $k = k_o = iK_o$ we have by Cauchy's Theorem:

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \psi(k,r)\, dk &= \text{Residue of } \psi(k,r) \text{ at } k = k_o \\ &= \lim_{k \to k_o} (k - k_o)\, \psi(k,r) \end{aligned} \tag{12}$$

Expanding $f(k)$ in powers of $k - k_o$ we obtain

$$\text{Residue} = \frac{\tan^{-1} k_o\, e^{ik_o r}}{\left.\frac{\partial f(k)}{\partial k}\right]_{k=k_o}} = \frac{\tan^{-1} k_o\, e^{ik_o r}}{-c \frac{\partial}{\partial k_o}\left(\frac{\tan^{-1} k_o}{k_o}\right)} \tag{13}$$

Differentiating $1 - \frac{c \tan^{-1} k_o}{k_o} = 0$ with respect to c gives:

$$-c \frac{\partial}{\partial k_o}\left(\frac{\tan^{-1} k_o}{k_o}\right) = \frac{\tan^{-1} k_o}{k_o \frac{\partial k_o}{\partial c}} \tag{14}$$

Thus:

$$\text{Residue} = k_o \frac{\partial k_o}{\partial c} e^{ik_o r} = -\frac{1}{2} \frac{\partial K_o^2}{\partial c} e^{-K_o r} \tag{15}$$

(12) then becomes:

$$\frac{1}{2\pi i}\int_C \psi(k,r)\, dk = -\frac{1}{2}\frac{\partial K_o^2}{\partial c} e^{-K_o r} \tag{16}$$

Now:

$$\frac{1}{2\pi i}\int_C \psi(k,r)\, dk = \sum_{i=1}^{6}\frac{1}{2\pi i}\int_{C_i}\psi(k,r)\, dk \tag{17}$$

$$\text{For } R\to\infty,\ \psi(k,r)\to \pm\frac{\pi}{2} \quad \text{on} \begin{Bmatrix} C_2 \\ C_6 \end{Bmatrix} \tag{18}$$

Therefore:

$$\frac{1}{2\pi i}\int_{C_2+C_6}\psi(k,r)\, dk \to \frac{1}{4i}\left\{\int_R^{iR} e^{ikr}\, dk - \int_{iR}^{R} e^{ikr}\, dk\right\}$$

$$= \frac{\cos Rr}{2r} - \frac{1}{2r} e^{-Rr} \xrightarrow[R\to\infty]{} \frac{\cos Rr}{2r} \tag{19}$$

Since

$$I(r) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\psi(k,r)\, dk = \lim_{R\to\infty}\left\{\frac{1}{2\pi i}\int_{-R}^{R}\psi(k,r)\, dk\right.$$

$$\left. + \frac{1}{2\pi i}\left(\int_R^{\infty} + \int_{-\infty}^{-R}\right)\psi(k,r)\, dk\right\} \tag{20}$$

$$= \lim_{R\to\infty}\left\{\frac{1}{2\pi i}\int_{C_1}\psi(k,r)\, dk + \frac{\cos Rr}{2r}\right\}$$

(where we interpret oscillating integrals in the sense of Cesaro summability).

We have, on noting that $\lim_{\epsilon\to 0}\int_{C_4} = 0$ since $k = i$ is a branch point but not a pole,

$$I(r) = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \left\{ -\frac{1}{2}\frac{\partial K_o^2}{\partial c} e^{-K_o r} - \frac{1}{2\pi i}\int_{C_3+C_5} \psi(k,r)\,dk \right\} \tag{21}$$

Using (8) we have:

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} -\frac{1}{2\pi i}\int_{C_3+C_5} \psi(k,r)\,dk$$

$$\begin{aligned} &= -\frac{1}{2\pi i}\int_{\infty}^{1} \left\{ i\tanh^{-1}\frac{1}{s} + \frac{\pi}{2} \right\} e^{-sr}\,i\,ds \\ &\quad - \frac{1}{2\pi i}\int_{1}^{\infty} \frac{\left\{ i\tanh^{-1}\frac{1}{s} - \frac{\pi}{2}\right\} e^{-sr}\,i\,ds}{\left\{ 1 - \frac{c\tanh^{-1}\frac{1}{s}}{s} - \frac{c\pi i}{2s} \right\}} \end{aligned} \tag{22}$$

$$= \frac{1}{2}\int_{1}^{\infty} \frac{e^{-sr}\,ds}{\left\{ \left(1 - \frac{c}{s}\tanh^{-1}\frac{1}{s}\right)^2 + \frac{c^2\pi^2}{4s^2} \right\}}$$

Changing the integration variable to $\mu = 1/s$ gives:

$$I(r) = -\frac{1}{2}\frac{\partial K_o^2}{\partial c} e^{-K_o r} + \frac{1}{2}\int_{0}^{1} e^{-r/\mu}\, g(c,\mu)\,\frac{d\mu}{\mu^2} \tag{23}$$

where $g(c,\mu)$ is that defined in (14-31).

Combining (1) and (23) yields

$$\rho(r) = \frac{1}{4\pi r}\left\{ -\frac{\partial K_o^2}{\partial c} e^{-K_o r} + \int_{0}^{1} g(c,\mu)\, e^{-r/\mu}\,\frac{d\mu}{\mu^2} \right\} \tag{24}$$

2. $c > 1$

In this case the two poles of $\psi(k,r)$ lie on the real axis at $\pm k_o$ where k_o is the positive real root of

$$1 - c\,\frac{\tan^{-1} k_o}{k_o} = 0 \qquad (25)$$

Without a prescription as to what is to be done at the poles the integral in (2) giving $\rho(r)$ is undefined. Different prescriptions such as going below or above the poles or taking the principal part of the integral will give different solutions. The lack of uniqueness as compared with the case $c < 1$ arises from the use of Fourier Transforms. This method automatically selects out the quadratically integrable solutions. For $c < 1$ this solution is unique. For $c > 1$ none of the solutions are strictly speaking quadratically integrable. They are all oscillatory.

The solution which is identical to (24) on replacing K_o by $-ik_o$ is obtained by choosing the path C' (Figure 35) for the integral I.

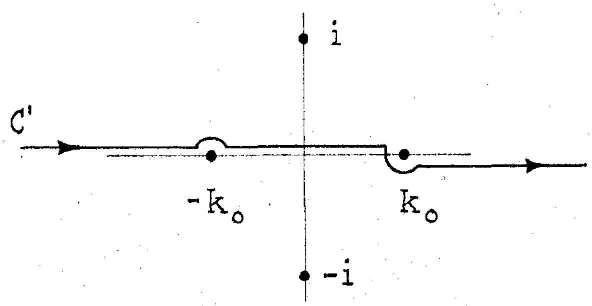

Fig. 35

This solution describes an outgoing wave in the time dependent problem. An incoming wave is obtained by integrating below $-k_o$ and above k_o. Standing waves can be constructed by a superposition of these. It may be noted that all these solutions are obtained from that found with the path C' by the addition of an infinite medium solution.

The analysis, being identical with the case $c < 1$, will be omitted.

3. $c = 1$

The result is obtained most simply by passing to the limit in the formula for $c < 1$ (equation 24) or in the formula for $c > 1$.